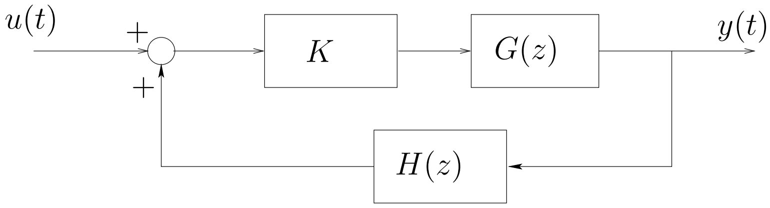
System Identification
Homework #1 on Time Series Prediction

Consider the stochastic process $y(t)$ generated as in following figure:



where

$$G(z) = \frac{z - \frac{3}{4}}{z - \frac{1}{2}} \ , \quad H(z) = \frac{1}{z} \ ,$$

K is a real constant and $u(t)$ is a Gaussian white process with zero mean and variance equal to 8.

1. Find the values of K for which $y(t)$ is asymptotically stationary.

   Hint: a second-order polynomial $P(x) = a_2 x^2 + a_1 x + a_0$ *has all the roots with modulus strictly less than 1 if and only if:* $P(1) > 0$, $P(-1) > 0$, $|a_2| > |a_0|$.

2. Find the spectrum $\Phi_y(z)$ of $y(t)$ and write the equation of $y(t)$ in the form of a MA, AR or ARMA process.

3. Set $K = 1$ and simulate a realization of process $y(t)$, for $t = 1, \ldots, N$, with $N = 1000$.

4. By using the realization obtained in item 3, compute the 2-step-ahead predictor $\hat{y}(t+2|t)$ and the corresponding sample mean square error. Compare the true time series $y(t)$ with the predicted one.

5. Compute the theoretical MSE of the Wiener predictor and compare it to the sample one computed in the previous item. Find a meaningful numerical procedure to improve the sample estimate of the MSE.

Solution of homework on Time Series Prediction

1. The transfer function from $u(t)$ to $y(t)$ is equal to

$$G(z) = \frac{K\,G(z)}{1 - K\,G(z)\,H(z)} = \frac{Kz\left(z - \frac{3}{4}\right)}{z^2 - \left(K + \frac{1}{2}\right)z + \frac{3}{4}K}.$$

The poles of the system are

$$p_{1,2} = \frac{K + \frac{1}{2} \pm \sqrt{K^2 - 2K + \frac{1}{4}}}{2}$$

and by imposing $|p_{1,2}| < 1$ one gets: $-\frac{6}{7} < K < \frac{4}{3}$.

2. The spectrum of $y(t)$ is given by

$$\begin{aligned}
\Phi_y(z) &= G(z)\,G(z^{-1})\,\sigma_u^2 = \frac{Kz\left(z - \frac{3}{4}\right)}{z^2 - \left(K + \frac{1}{2}\right)z + \frac{3}{4}K}\,\frac{Kz^{-1}\left(z^{-1} - \frac{3}{4}\right)}{z^{-2} - \left(K + \frac{1}{2}\right)z^{-1} + \frac{3}{4}K}\,8 \\
&= \frac{1 - \frac{3}{4}z^{-1}}{1 - \left(K + \frac{1}{2}\right)z^{-1} + \frac{3}{4}Kz^{-2}}\,\frac{1 - \frac{3}{4}z}{1 - \left(K + \frac{1}{2}\right)z + \frac{3}{4}Kz^{2}}\,8K^2.
\end{aligned}$$

Therefore, $y(t)$ is an ARMA(2,1) process, whose equation can be written as

$$y(t) - \left(K + \frac{1}{2}\right)y(t-1) + \frac{3}{4}Ky(t-2) = e(t) - \frac{3}{4}e(t-1)$$

where $e(t)$ is a white process with zero mean and variance $\sigma_e^2 = 8K^2$.

3.-4. See file `homework_timeseries_solution.m`

5. From the theory, one can compute the MSE of the Wiener predictor by applying the polynomial division

$$\begin{array}{l|l}
1 - \frac{3}{4}z^{-1} & 1 - \frac{3}{2}z^{1} + \frac{3}{4}z^{-2} \\ \cline{2-2}
1 - \frac{3}{2}z^{-1} + \frac{3}{4}z^{-2} & 1 + \frac{3}{4}z^{-1} \\ \hline
\quad \frac{3}{4}z^{-1} - \frac{3}{4}z^{-2} & \\
\quad \frac{3}{4}z^{-1} - \frac{9}{8}z^{-2} + \frac{9}{16}z^{-3} & \\ \hline
\qquad \frac{3}{8}z^{-2} - \frac{9}{16}z^{-3} &
\end{array}$$

which yields the Wiener predictor

$$\hat{y}(t+2|t) = \frac{\frac{3}{8} - \frac{9}{16}z^{-1}}{1 - \frac{3}{4}z^{-1}} y(t)$$

whose MSE is given by

$$MSE_{Wiener} = \sigma_e^2 \left(1 + q_1^2\right) = 8 \left(1 + \frac{9}{16}\right) = 12.5.$$

A possible alternative to the theoretical approach is to repeat steps 3.-4. for a large number of realizations (say, 1000) and then average the sample MSE values obtained by the Wiener predictor. The resulting average approaches the theoretical MSE value as the number of realizations increase (see `homework_timeseries_solution.m`).