1.I) Funzione di trasferimento da $u(t)$ a $y(t)$

$$P(s) = [1 \ \ 0] \begin{bmatrix} s & -1 \\ 1 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{s^2+s+1}$$

$$W(s) = \frac{G(s)P(s)}{1+C(s)G(s)P(s)} = \frac{\frac{k}{s}\,\frac{1}{s^2+s+1}}{1+\frac{1}{s+3}\,\frac{k}{s}\,\frac{1}{s^2+s+1}} =$$

$$= \frac{k(s+3)}{s(s+3)(s^2+s+1)+k} = \frac{k(s+3)}{s^4+4s^3+4s^2+3s+k}$$

1.II)

$$\begin{array}{c|ccc} 4 & 1 & 4 & k \\ 3 & 4 & 3 & \\ 2 & \frac{13}{4} & k & \\ 1 & \dfrac{\frac{39}{4}-4k}{\frac{13}{4}} & & \longrightarrow \quad \dfrac{39-16k}{4} > 0 \\ 0 & k & & \longrightarrow \quad k > 0 \end{array}$$

$$\Longrightarrow \quad k \in \left(0, \frac{39}{16}\right)$$

1.III)

$$W(j) = \frac{3+j}{1-4j-4+3j+1} = \frac{3+j}{-2-j} = \frac{(3+j)(-2+j)}{5} =$$

$$= \frac{-7+j}{5} = \frac{\sqrt{50}}{5}\, e^{j\left(-\operatorname{atan}\left(\frac{1}{7}\right)+\pi\right)} = \sqrt{2}\, e^{-j\left(\operatorname{atan}\left(\frac{1}{7}\right)-\pi\right)}$$

$$y_p(t) = 2\cos\left(t - \operatorname{atan}\left(\frac{1}{7}\right)+\pi\right)$$

2.I)

$$x_1(k+1) = \frac{1}{2}x_1(k) + \frac{1}{4}x_2(k)$$
$$x_2(k+1) = \frac{1}{2}x_1(k) + \frac{1}{2}x_2(k) + \frac{3}{4}x_3(k)$$
$$x_3(k+1) = \frac{1}{4}x_2(k) + \frac{1}{4}x_3(k)$$

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{3}{4} \\ 0 & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

2.II)

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda-\frac{1}{2} & -\frac{1}{4} & 0 \\ -\frac{1}{2} & \lambda-\frac{1}{2} & -\frac{3}{4} \\ 0 & -\frac{1}{4} & \lambda-\frac{1}{4} \end{bmatrix} =$$

$$= \left(\lambda-\frac{1}{2}\right)\left[\left(\lambda-\frac{1}{2}\right)\left(\lambda-\frac{1}{4}\right)-\frac{3}{16}\right] + \frac{1}{2}\left(-\frac{1}{4}\right)\left(\lambda-\frac{1}{4}\right)$$

$$= \left(\lambda-\frac{1}{2}\right)\left(\lambda^2-\frac{3}{4}\lambda-\frac{1}{16}\right) - \frac{1}{8}\left(\lambda-\frac{1}{4}\right) =$$

$$= \lambda^3 - \frac{5}{4}\lambda^2 + \frac{3}{16}\lambda + \frac{1}{16} = (\lambda-1)\left(\lambda^2-\frac{1}{4}\lambda-\frac{1}{16}\right)$$

autovalori $\lambda = 1$ $\quad \lambda = \dfrac{\frac{1}{4} \pm \sqrt{\frac{5}{16}}}{2} = \frac{1}{8} \pm \frac{\sqrt{5}}{8}$

modi: $\mathbb{1}(k)$ limitato non convergente

$\left(\frac{1+\sqrt{5}}{8}\right)^k, \left(\frac{1-\sqrt{5}}{8}\right)^k$ convergenti

2.III)

$x(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\quad X_1(z) = [1 \ \ 0 \ \ 0] \begin{bmatrix} z-\frac{1}{2} & -\frac{1}{4} & 0 \\ -\frac{1}{2} & z-\frac{1}{2} & -\frac{3}{4} \\ 0 & -\frac{1}{4} & z-\frac{1}{4} \end{bmatrix}^{-1} z \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$= \frac{\left[\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)-\frac{3}{16}\right]z}{(z-1)\left(z^2-\frac{1}{4}z-\frac{1}{16}\right)}$$

$$\lim_{k\to+\infty} x_1(k) = \lim_{z\to 1}(z-1)X_1(z) = \frac{\frac{3}{16}}{\frac{11}{16}} = \frac{3}{11}$$

3.I)

$$A = \begin{bmatrix} -1 & -1 \\ 1 & -\alpha \end{bmatrix} \qquad \det(\lambda I - A) = (\lambda+1)(\lambda+\alpha)+1 =$$
$$= \lambda^2 + (1+\alpha)\lambda + (1+\alpha)$$

modi convergenti $\rightarrow$ $1+\alpha > 0$

modi aperiodici $\rightarrow$ $\Delta = (1+\alpha)^2 - 4(1+\alpha) > 0$

$$\Delta = (1+\alpha)(\alpha-3) > 0 \qquad \Longrightarrow \qquad \boxed{\alpha > 3}$$

3.II)

$$G(s) = [0 \quad 1] \begin{bmatrix} s+1 & 1 \\ -1 & s+\frac{7}{2} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{s+2}{s^2 + \frac{9}{2}s + \frac{9}{2}}$$

$$\text{poli} = \frac{-\frac{9}{2} \pm \sqrt{\frac{81}{4} - 18}}{2} = -\frac{9}{4} \pm \frac{3}{4} = \begin{cases} -\frac{3}{2} \\ -3 \end{cases}$$

$$Y_f(s) = \frac{s+2}{s\left(s+\frac{3}{2}\right)(s+3)} = \frac{R_1}{s} + \frac{R_2}{s+\frac{3}{2}} + \frac{R_3}{s+3}$$

$$R_1 = \lim_{s\to 0} s\, Y_f(s) = \frac{4}{9} \qquad R_2 = \lim_{s \to -\frac{3}{2}} \left(s+\frac{3}{2}\right) Y_f(s) = -\frac{2}{9}$$

$$R_3 = \lim_{s\to -3} (s+3)\, Y_f(s) = -\frac{2}{9}$$

$$y_f(t) = \left\{ \frac{4}{9} - \frac{2}{9} e^{-\frac{3}{2}t} - \frac{2}{9} e^{-3t} \right\} \cdot 1\!\!1(t)$$

3.III)

$$Y_\ell(s) = [0 \quad 1] \begin{bmatrix} s+1 & 1 \\ -1 & s+\frac{7}{2} \end{bmatrix}^{-1} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} =$$

$$= \frac{x_1(0) + (s+1)\, x_2(0)}{(s+3)\left(s+\frac{3}{2}\right)}$$

$$\Rightarrow \quad x_1(0) + \left(-\frac{3}{2}+1\right) x_2(0) = 0 \qquad x_1(0) = \frac{1}{2} x_2(0)$$

$$\Rightarrow \quad x(0) = L \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

4.I) $G(z) = \dfrac{Y_f(z)}{\dfrac{z}{z-1}}$ con $Y_f(z) = \dfrac{z}{z-\frac{3}{4}} - \dfrac{z}{z-\frac{1}{4}}$

$$\Rightarrow G(z) = \frac{z-1}{z} \cdot \frac{\frac{1}{2}z}{\left(z-\frac{3}{4}\right)\left(z-\frac{1}{4}\right)} = \frac{\frac{1}{2}z - \frac{1}{2}}{z^2 - z + \frac{3}{16}}$$

i/o: $y(k+2) - y(k+1) + \frac{3}{16} y(k) = \frac{1}{2} u(k+1) - \frac{1}{2} u(k)$

4.II) $y_{imp}(k) = \mathcal{Z}^{-1}\left[G(z)\right] = \dfrac{R_1}{z-\frac{3}{4}} + \dfrac{R_2}{z-\frac{1}{4}}$

$$\begin{cases} R_1 + R_2 = \frac{1}{2} \\ -\frac{1}{4}R_1 - \frac{3}{4}R_2 = -\frac{1}{2} \end{cases} \Rightarrow \begin{cases} R_2 = \frac{1}{2} - R_1 \\ -\frac{1}{4}R_1 - \frac{3}{8} + \frac{3}{4}R_1 = -\frac{1}{2} \end{cases} \Rightarrow \begin{matrix} R_1 = -\frac{1}{4} \\ R_2 = \frac{3}{4} \end{matrix}$$

$$y_{imp}(k) = \left\{ \frac{3}{4}\left(\frac{1}{4}\right)^{k-1} - \frac{1}{4}\left(\frac{3}{4}\right)^{k-1} \right\} \cdot \mathbb{1}(k-1)$$

4.III) Affinché $\lim\limits_{k\to+\infty}$ $\exists$ finito occorre che

$U(z) = \gamma \cdot \dfrac{z}{(z-1)^2}$ ($G(z)$ ha uno zero in $z=1$)

Si ha così:

$$\lim_{k\to+\infty} y_f(k) = \lim_{z\to1} (z-1) \cdot \frac{\frac{1}{2}(z-1)}{z^2 - z + \frac{3}{16}} \frac{\gamma z}{(z-1)^2} =$$

$$= \gamma \frac{\frac{1}{2}}{\frac{3}{16}} = \frac{16}{6}\gamma \Rightarrow \gamma = 6$$

Quindi $u(k) = 6k \cdot \mathbb{1}(k)$