1.I) $P(z) = \dfrac{G_1(z)}{1+\frac{1}{z}G_1(z)}$ $\quad W(z) = \dfrac{P(z)\,G_2(z)}{1+\frac{1}{z}P(z)\,G_2(z)}$

$$P(z) = \frac{\frac{z}{z-\frac{1}{2}}}{1+\frac{1}{z}\frac{z}{z-\frac{1}{2}}} = \frac{z}{z+\frac{1}{2}}$$

$$W(z) = \frac{\frac{z^2}{(z+\frac{1}{2})(z-1)}}{1+\frac{z}{(z+\frac{1}{2})(z-1)}} = \frac{z^2}{z^2+\frac{1}{2}z-\frac{1}{2}} = \frac{z^2}{(z-\frac{1}{2})(z+1)}$$

1.II) $y_{imp}(k) = \mathcal{Z}^{-1}[W(z)]$

$$\frac{W(z)}{z} = \frac{R_1}{z-\frac{1}{2}} + \frac{R_2}{z+1} \qquad \begin{cases} R_1+R_2 = 1 \\ R_1 - \frac{1}{2}R_2 = 0 \end{cases} \quad \begin{matrix} R_1 = \frac{1}{3} \\ R_2 = \frac{2}{3} \end{matrix}$$

$$y_{imp}(k) = \left\{ \frac{1}{3}\left(\frac{1}{2}\right)^k + \frac{2}{3}(-1)^k \right\} \cdot 1\!\!1(k)$$

1.III) $Y_f(z) = W(z) \cdot U(z)$

$u(k)$ durata finita $\Rightarrow$ $U(z) = \dfrac{N_u(z)}{z^h}$

Affinché $y_f(k)$ non converga a zero è sufficiente che $N_u(z)$ non cancelli il polo in $z=-1$.

Ad esempio se si sceglie $U(z) = \dfrac{z-\frac{1}{2}}{z}$ (cioè $u(k) = \delta(k) - \frac{1}{2}\delta(k-1)$) si ottiene

$$Y_f(z) = \frac{z^2}{(z-\frac{1}{2})(z+1)} \cdot \frac{z-\frac{1}{2}}{z} = \frac{z}{z+1} \Rightarrow y_f(k) = (-1)^k \cdot 1\!\!1(k)$$

(... ma andava bene anche $u(k) = \delta(k)$!)

2.I) $x_1(t) = y(t)$ $x_2(t) = \dot{y}(t)$ $x_3(t) = \ddot{y}(t)$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ k-1 & -2k & -1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad C = [1 \quad 0 \quad 0]$$

$$D = [0]$$

2.II) Polinomio caratteristico $p(s) = s^3 + s^2 + 2ks + (1-k$

3	1	2k
2	1	1-k
1	3k-1	
0	1-k	

$$\begin{cases} 3k-1 > 0 \\ 1-k > 0 \end{cases} \Rightarrow \begin{matrix} k > \frac{1}{3} \\ k < 1 \end{matrix}$$

Modi convergenti per $k \in \left(\frac{1}{3}, 1\right)$

2.III) $G(s) = C(sI-A)^{-1}B + D =$

$$= [1 \quad 0 \quad 0] \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ \frac{1}{2} & 1 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{s^3 + s^2 + s + \frac{1}{2}}$$

$$\omega = 1 \Rightarrow G(j) = \frac{1}{-1 + \frac{1}{2}} = -2$$

$|G(j)| = 2$ $\quad \measuredangle G(j) = \pi$

$$y_{perm}(t) = 2 \cdot 10 \cos(t + \pi) = -20 \cos(t)$$

3.I) $x_i(k)$: clienti dell'albergo i nell'anno k

$$x_1(k+1) = \frac{1}{2}x_1(k) + \frac{1}{4}x_2(k) + u_1(k)$$
$$x_2(k+1) = \frac{1}{4}x_1(k) + \frac{1}{2}x_2(k) + u_2(k)$$
$$y(k) = x_1(k) + x_2(k)$$

3.II) $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}$ $\quad \det(\lambda I - A) = \lambda^2 - \lambda + \frac{3}{16} = $
$$= \left(\lambda - \frac{1}{4}\right)\left(\lambda - \frac{3}{4}\right)$$

modi: $\left(\frac{1}{4}\right)^k$, $\left(\frac{3}{4}\right)^k$, entrambi convergenti

3.III) $u(k) = \begin{bmatrix} N \\ 2N \end{bmatrix} \cdot 1\!\!1(k)$

$$X_f(z) = (zI - A)^{-1}B \cdot U(z) = (zI-A)^{-1}B \cdot \begin{bmatrix} N \\ 2N \end{bmatrix} \frac{z}{z-1}$$

Con $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\lim_{k \to +\infty} x_f(k) = \lim_{z \to 1} (z-1) X_f(z) = (I-A)^{-1}B \begin{bmatrix} N \\ 2N \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{2} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} N = \frac{16}{3} \begin{bmatrix} 1/2 & 1/4 \\ 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} N =$$

$$= \begin{bmatrix} \frac{16}{3} N \\ \frac{20}{3} N \end{bmatrix}$$

Ponendo $\frac{20}{3}N = 100$ si ottiene $N = 15$ e

$$\lim_{k \to +\infty} x_1(k) = \frac{16}{3} \cdot 15 = 80$$

4.I) $A = \begin{bmatrix} -1 & \alpha \\ -1 & -5 \end{bmatrix}$ $\det(\lambda I - A) = \det\begin{pmatrix} \lambda+1 & -\alpha \\ 1 & \lambda+5 \end{pmatrix} =$

$= \lambda^2 + 6\lambda + 5 + \alpha$

$\Delta = 36 - 20 - 4\alpha = 16 - 4\alpha$

modi aperiodici per $\alpha \leq 4$, pseudoperiodici per $\alpha > 4$

se $\alpha > -5$ $\rightarrow$ 2 permanenze $\rightarrow$ 2 modi convergenti

se $\alpha = -5$ $\rightarrow$ $\lambda^2 + 6\lambda = \lambda(\lambda+6)$ $\rightarrow$ 1 modo convergente
1 modo limitato (gradino)

se $\alpha < -5$ $\rightarrow$ 1 permanenza $\rightarrow$ 1 modo convergente
1 modo divergente

4.II) $G(s) = C(sI-A)^{-1}B + D =$

$= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s+1 & -4 \\ +1 & s+5 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \dfrac{s+5}{s^2+6s+9}$

$Y_f(s) = \dfrac{s+5}{s^2+6s+9} \cdot \dfrac{1}{s} = \dfrac{R_1}{s} + \dfrac{R_{2,1}}{s+3} + \dfrac{R_{2,2}}{(s+3)^2}$

$R_1(s+3)^2 + R_{2,1}(s^2+3s) + R_{2,2}\, s = s+5$

$\Rightarrow R_1 = \frac{5}{9} \quad R_{2,1} = -\frac{5}{9} \quad R_{2,2} = -\frac{2}{3}$

$y_f(t) = \left\{ \frac{5}{9} - \frac{5}{9} e^{-3t} - \frac{2}{3} t e^{-3t} \right\} \cdot 1\!\!1(t)$

4.III) $Y_\ell(s) = C(sI-A)^{-1}x(0) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s+1 & -\alpha \\ +1 & s+5 \end{bmatrix}^{-1} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$

$= \dfrac{(s+4)x_1(0) + (s+1+\alpha)x_2(0)}{s^2+6s+5+\alpha} = \dfrac{1}{s}$

$(s^2+4s)x_1(0) + (s^2 + (1+\alpha)s)\, x_2(0) = s^2+6s+5+\alpha$

$\begin{cases} 5+\alpha = 0 \\ 4x_1(0) + (1+\alpha)x_2(0) = 6 \\ x_1(0) + x_2(0) = 1 \end{cases} \Rightarrow \quad \alpha = -5, \quad x_1(0) = \frac{5}{4} \quad x_2(0) = -\frac{1}{4}$