1.I
$$x_1(k+1) = x_2(k)$$
$$x_2(k+1) = x_3(k)$$
$$x_3(k+1) = \frac{1}{4}x_2(k) + u(k) \qquad y(k) = x_1(k)$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & \frac{1}{4} & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad C = [1 \quad 0 \quad 0] \quad D = \emptyset$$

1.II
$$Y_\ell(z) = C(zI - A)^{-1} z\, x(0) = [1 \quad 0 \quad 0] \begin{bmatrix} z & -1 & 0 \\ 0 & z & -1 \\ 0 & -\frac{1}{4} & z \end{bmatrix}^{-1} z \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} =$$
$$= \frac{z\left\{(z^2 - \frac{1}{4})x_1(0) + z\,x_2(0) + x_3(0)\right\}}{z\left(z^2 - \frac{1}{4}\right)}$$
$$\mathcal{Z}\left[\left(\frac{1}{2}\right)^{k-1} \mathbb{1}(k-1)\right] = \frac{1}{z - \frac{1}{2}}$$
$$\left(z^2 - \frac{1}{4}\right)x_1(0) + z\,x_2(0) + x_3(0) = z + \frac{1}{2}$$
$$\Rightarrow \quad x_1(0) = \emptyset \qquad x_2(0) = 1 \qquad x_3(0) = \frac{1}{2}$$

1.III
$$G(z) = C(zI - A)^{-1}B + D = \frac{1}{z\left(z^2 - \frac{1}{4}\right)}$$
$$y_{imp}(k) = \mathcal{Z}^{-1}\left[\frac{1}{z\left(z - \frac{1}{2}\right)\left(z + \frac{1}{2}\right)}\right]$$
$$G(z) = \frac{R_1}{z} + \frac{R_2}{z - \frac{1}{2}} + \frac{R_3}{z + \frac{1}{2}} = \frac{-4}{z} + \frac{2}{z - \frac{1}{2}} + \frac{2}{z + \frac{1}{2}}$$
$$y_{imp}(k) = \left\{2 \cdot \left(\frac{1}{2}\right)^{k-1} + 2\left(-\frac{1}{2}\right)^{k-1} - 4\,\delta(k-1)\right\} \cdot \mathbb{1}(k-1)$$

2.I $\quad W(s) = \dfrac{G(s)+K}{1+(G(s)+k)H(s)} = \dfrac{\frac{12}{s-2}+k}{1+\frac{1}{s}\left(\frac{12}{s-2}+k\right)} =$

$$= \frac{s(12+k(s-2))}{s^2-2s+12+k(s-2)} = \frac{s[ks+12-2k]}{s^2+(k-2)s+12-2k}$$

2.II

Regola di Cartesio: $\begin{cases} k-2>0 \\ 12-2k>0 \end{cases} \rightarrow k \in (2,6)$

$k=2$: $\quad s^2+8=0 \rightarrow$ modi non convergenti

$k=6$: $\quad W(s) = \dfrac{s[6s]}{s^2+4s} = \dfrac{6s}{s+4} \rightarrow$ un solo modo convergente!

$\Rightarrow k \in (2,6]$

2.III $\quad W(s) = \dfrac{4s(s+1)}{s^2+2s+4}$

$$|W(j\omega)| = \left|\frac{4j\omega(1+j\omega)}{4-\omega^2+2j\omega}\right| = 2\sqrt{5}$$

$$\frac{4\omega\sqrt{1+\omega^2}}{\sqrt{(4-\omega^2)^2+4\omega^2}} = 2\sqrt{5}$$

$$16\,\omega^2(1+\omega^2) = 20\,(16+\omega^4-4\omega^2)$$

$$\omega^4-24\,\omega^2+80=0$$

$$\omega^2 = \begin{cases} 20 \\ 4 \end{cases} \Rightarrow \omega = \begin{cases} 2\sqrt{5} \\ 2 \end{cases}$$

3.I

$$x_1(k+1) = 0.8\,x_1(k) + 0.1\,x_2(k) + u(k)$$
$$x_2(k+1) = 0.1\,x_1(k) + 0.8\,x_2(k)$$
$$y(k) = x_1(k) + x_2(k)$$

3.II

$$x(0) = \begin{bmatrix} 2.5 \\ 2.5 \end{bmatrix}$$ Malintenti

$$Y_\ell(z) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} z-0.8 & -0.1 \\ -0.1 & z-0.8 \end{bmatrix}^{-1} z \begin{bmatrix} 2.5 \\ 2.5 \end{bmatrix} =$$

$$= 2.5\,z \frac{2(z-0.8)+0.2}{(z-0.8)^2-0.01} = 5z\frac{z-0.7}{z^2-1.6z+0.63} =$$

$$= 5z\frac{z-0.7}{(z-0.7)(z-0.9)} = 5\frac{z}{z-0.9}$$

$$y_\ell(k) = 5\cdot(0.9)^k\cdot \mathbb{1}(k)$$

3.III

$$u(k) = \mathbb{1}(k)$$

$$X_f(z) = \left[(zI-A)^{-1}B\right]\cdot U(z) =$$

$$= \begin{bmatrix} z-0.8 & -0.1 \\ -0.1 & z-0.8 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}\cdot \frac{z}{z-1} =$$

$$= \frac{\begin{bmatrix} z-0.8 \\ 0.1 \end{bmatrix}\cdot z}{(z^2-1.6z+0.63)(z-1)}$$

$$\lim_{k\to+\infty} x_f(k) = \lim_{z\to 1}(z-1)X_f(z) = \begin{bmatrix} \frac{0.2}{0.03} \\ \frac{0.1}{0.03} \end{bmatrix}$$

Percentuale di popolazione:

in x_1 : $2/3 \simeq 66.7\%$

in x_2 : $1/3 \simeq 33.3\%$

4.I

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -6 & -4 \end{bmatrix}$$

$$\det(\lambda I - A) = \lambda \cdot (\lambda^2 + 4\lambda + 6) = \emptyset$$

$$\lambda_1 = \emptyset \qquad \lambda_{2,3} = -2 \pm j\sqrt{2}$$

modi: $1(t)$, $e^{-2t}\cos(\sqrt{2}t)$, $e^{-2t}\sin(\sqrt{2}t)$

aperiodico, limitato non convergente; pseudoperiodi, convergenti

4.II

$$Y_f(s) = \left[C(sI-A)^{-1}B + D\right] \cdot \frac{1}{s} =$$

$$= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 0 & 6 & s+4 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} \cdot \frac{1}{s} =$$

$$= \frac{4}{s^2(s^2+4s+6)} = \frac{\gamma_0}{s} + \frac{\gamma_1}{s^2} + \frac{\gamma_2(s+2)}{(s+2)^2+2} + \frac{\gamma_3\sqrt{2}}{(s+2)^2+2}$$

$$4 = \gamma_0(s^3+4s^2+6s) + \gamma_1(s^2+4s+6) + \gamma_2(s^3+2s^2) + \gamma_3\sqrt{2}\,s^2$$

$$\begin{cases} \gamma_0 + \gamma_2 = 0 \\ 4\gamma_0 + \gamma_1 + 2\gamma_2 + \sqrt{2}\,\gamma_3 = 0 \\ 6\gamma_0 + 4\gamma_1 = 0 \\ 6\gamma_1 = 4 \end{cases} \Rightarrow \quad \gamma_1 = \frac{2}{3}, \quad \gamma_0 = -\frac{4}{9}, \quad \gamma_2 = \frac{4}{9}, \quad \gamma_3 = \frac{2}{9\sqrt{2}} = \frac{\sqrt{2}}{9}$$

$$y_f(t) = \left\{ \frac{2}{3}t - \frac{4}{9} + \frac{4}{9}e^{-2t}\cos(\sqrt{2}t) + \frac{\sqrt{2}}{9}e^{-2t}\sin(\sqrt{2}t) \right\} 1(t)$$

4.III

$$U(s) = \frac{1}{s+1} - K\frac{1}{s+2} = \frac{s+2-K(s+1)}{(s+1)(s+2)}$$

Affinché $y_f(t) \to \emptyset$, occorre cancellare il polo in $s = \emptyset$ di $G(s)$, quindi: $s+2-K(s+1) = \alpha \cdot s$

$\Rightarrow 2 - K = \emptyset \Rightarrow K = 2$