1.I $$W(s) = \frac{C(s)\cdot G(s)}{1 + C(s)G(s)H(s)} = \frac{2\cdot \frac{1}{s+1}}{1 + 2\cdot \frac{1}{s+1}\cdot\frac{1}{s+1}} =$$

$$= \frac{2(s+1)}{(s+1)^2 + 2} = \frac{2s+2}{s^2+2s+3}$$

Poiché i poli di $W(s)$ hanno parte reale < 0 si può applicare il teorema del valore finale

$$\lim_{t\to+\infty} y_f(t) = \lim_{s\to 0} \cancel{s}\cdot W(s)\cdot \gamma \cdot \frac{1}{\cancel{s}} = W(0)\cdot\gamma = 1$$

$$W(0) = \frac{2}{3} \implies \gamma = \frac{3}{2}.$$

1.II $$y_f(t) = \mathcal{L}^{-1}[W(s)] = \mathcal{L}^{-1}\left[\frac{2s+2}{s^2+2s+3}\right]$$

poli: $-1 \pm j\sqrt{2} \to$ modi: $e^{-t}\cos(\sqrt{2}t)$, $e^{-t}\sin(\sqrt{2}t)$

$$W(s) = A\cdot\frac{s+1}{(s+1)^2+2} + B\,\frac{\sqrt{2}}{(s+1)^2+2} = \frac{2s+2}{(s+1)^2+2}$$

$$A(s+1) + B\sqrt{2} = 2s+2$$

$$As + A + \sqrt{2}B = 2s+2 \longrightarrow \begin{cases} A = 2 \\ A+\sqrt{2}B = 2 \end{cases} \qquad \begin{matrix} A=2 \\ B=0 \end{matrix}$$

$$y_f(t) = 2e^{-t}\cos(\sqrt{2}t)\cdot 1(t)$$

1.III $$|W(j\omega)| > \frac{1}{\sqrt{3}}$$

$$\left|\frac{2j\omega+2}{(j\omega)^2+2j\omega+3}\right| = 2\,\frac{|1+j\omega|}{|(3-\omega^2)+2j\omega|} = 2\cdot\frac{\sqrt{1+\omega^2}}{\sqrt{(3-\omega^2)^2+4\omega^2}}$$

$$4\cdot\frac{1+\omega^2}{(3-\omega^2)^2+4\omega^2} > \frac{1}{3}$$

$$12 + 12\omega^2 > \omega^4 - 6\omega^2 + 9 + 4\omega^2$$

$$\omega^4 - 14\omega^2 - 3 < 0$$

$$\omega^2 = x \quad \rightarrow \quad x^2 - 14x - 3 < 0$$

$$x_{1,2} = 7 \pm \sqrt{52} \quad \longrightarrow \quad x \in (\underbrace{7-\sqrt{52}}_{<0},\ 7+\sqrt{52})$$

$$\longrightarrow \quad \omega \in (\emptyset, \sqrt{7+\sqrt{52}})$$

2.I
$$\dot{x}_1(t) = 5x_2(t)$$
$$\dot{x}_2(t) = -5x_1(t) + kx_2(t) + u(t)$$
$$y(t) = x_1(t) + x_2(t)$$

$$A = \begin{bmatrix} 0 & 5 \\ -5 & k \end{bmatrix} \qquad \det(\lambda I - A) = \det\begin{pmatrix} \lambda & -5 \\ 5 & \lambda - k \end{pmatrix} = \lambda^2 - k\lambda + 25$$

Per Cartesio:
- $k<0$ modi convergenti
- $k=0$ modi limitati non convergenti
- $k>0$ modi divergenti

2.II
$$G(s) = C(sI-A)^{-1}B =$$
$$= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s & -5 \\ 5 & s-k \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{s+5}{s^2 - ks + 25}$$

$$k=0 \implies G(s) = \frac{s+5}{s^2+25} \longrightarrow$$ modi: $\cos(5t)$, $\sin(5t)$

Scegliendo $u(t) = \cos(5t)$ si ottiene
$Y_f(s) = \dfrac{*}{(s^2+25)^2}$ e quindi $y_f(t)$ contiene il modo divergente $t\cos(5t)$.

2. III $\quad y_e(t) = \sin(5t) \implies Y_e(s) = \dfrac{5}{s^2+25}$

$$Y_e(s) = C(sI-A)^{-1} \cdot x(\emptyset) =$$

$$= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s & -5 \\ 5 & s-k \end{bmatrix}^{-1} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} =$$

$$= \frac{\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s-k & 5 \\ -5 & s \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}}{s^2 - \underset{\substack{\parallel \\ \emptyset}}{k}s + 25} = \frac{5}{s^2+25}$$

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s x_1(0) + 5 x_2(0) \\ -5 x_1(0) + s x_2(0) \end{bmatrix} = 5$$

$$s(x_1(0) + x_2(0)) + 5x_2(0) - 5x_1(0) = 5$$

$$\begin{cases} x_1(0) + x_2(0) = \emptyset \\ x_2(0) - x_1(0) = 1 \end{cases} \qquad x_2(0) = \frac{1}{2} \quad x_1(0) = -\frac{1}{2}$$

$$x(0) = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

3.I $\quad g(k) = k \cdot \mathbb{1}(k)$

$$G(z) = \frac{z}{(z-1)^2}$$

3.II $\quad u(k) = u_0 \cdot \delta(k) + u_1 \delta(k-1) + u_2 \delta(k-2) \cdots u_h \delta(k-h)$

$$\hookrightarrow U(z) = u_0 + \frac{u_1}{z} + \frac{u_2}{z^2} + \cdots + \frac{u_h}{z^h} = \frac{N(z)}{z^h}$$

$$Y_f(z) = G(z) \cdot U(z) = \frac{z}{(z-1)^2} \cdot \frac{N(z)}{z^h}$$

$$U(z) = \frac{z-1}{z} \qquad u(k) = \{1, -1, 0, 0, \cdots\}$$

$$\Rightarrow Y_f(z) = \frac{1}{z-1} \quad \Rightarrow \quad y_f(k) = \mathbb{1}(k-1)$$

3.III $\quad G(z) = \dfrac{z}{z^2 - 2z + 1}$

i/o: $\quad y(k+2) - 2y(k+1) + y(k) = u(k+1)$

$$y(k+1) - 2\underbrace{y(k)}_{x_2(k)} + \underbrace{y(k-1)}_{x_1(k)} = u(k)$$

$$x_1(k+1) = x_2(k)$$
$$x_2(k+1) = 2x_2(k) - x_1(k) + u(k)$$
$$y(k) = x_2(k)$$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad C = (0 \quad 1) \quad D = [0]$$

4.I

$$x_1(k+1) = 0.25\, x_1(k) + 0.5\, x_2(k) + 0.5\, x_3(k)$$
$$x_2(k+1) = 0.375\, x_1(k) + 0.5\, x_2(k)$$
$$x_3(k+1) = 0.375\, x_1(k) + 0.5\, x_3(k)$$

4.II

$$A = \begin{bmatrix} 0.25 & 0.5 & 0.5 \\ 0.375 & 0.5 & 0 \\ 0.375 & 0 & 0.5 \end{bmatrix}$$

$$\det \begin{pmatrix} \lambda - 0.25 & -0.5 & -0.5 \\ -0.375 & \lambda - 0.5 & 0 \\ -0.375 & 0 & \lambda - 0.5 \end{pmatrix} =$$

$$(\lambda - 0.5)\{(\lambda - 0.25)(\lambda - 0.5) - 0.5 \cdot 0.375\} - 0.5 \cdot$$
$$0.375\,(\lambda - 0.5) =$$
$$= (\lambda - 0.5)\{\lambda^2 - 0.75\lambda + 0.125 - 0.5 \cdot 0.375 - 0.5 \cdot 0.375\} =$$
$$= (\lambda - 0.5)\{\lambda^2 - 0.75\lambda - 0.25\} =$$
$$= (\lambda - 0.5)(\lambda - 1)(\lambda + 0.25)$$

Modi: $\mathbb{1}(k)$, $(0.5)^k \cdot \mathbb{1}(k)$, $(-0.25)^k \cdot \mathbb{1}(k)$

4.III

$$x(0) = \begin{pmatrix} 100 \\ 0 \\ 0 \end{pmatrix} \qquad y(k) = x_1(k)$$

$$Y_e(z) = C\,(zI - A)^{-1} z \cdot x(0) =$$

$$= (1 \;\; 0 \;\; 0) \begin{pmatrix} z - 0.25 & -0.5 & -0.5 \\ -0.375 & z - 0.5 & 0 \\ -0.375 & 0 & z - 0.5 \end{pmatrix}^{-1} \cdot z \begin{pmatrix} 100 \\ 0 \\ 0 \end{pmatrix} =$$

$$= 100\, z \cdot \frac{(z-0.5)^2}{\cancel{(z-0.5)}(z-1)(z+0.25)}$$

$$\frac{Y_e(z)}{z} = \frac{100\,(z-0.5)}{(z-1)(z+0.25)} = \frac{R_1}{z-1} + \frac{R_2}{z+0.25}$$

$$R_1 = \lim_{z \to 1} (z-1)\frac{Y_e(z)}{z} = \frac{100 \cdot 0.5}{1.25} = \frac{50}{1.25}$$

$$R_2 = \lim_{z \to -0.25} (z+0.25)\frac{Y_e(z)}{z} = \frac{100\,(-0.75)}{-1.25} = \frac{75}{1.25}$$

$$y_e(k) = \frac{50}{1.25} \cdot \mathbb{1}(k) + \frac{75}{1.25} \cdot (-0.25)^k \cdot \mathbb{1}(k)$$