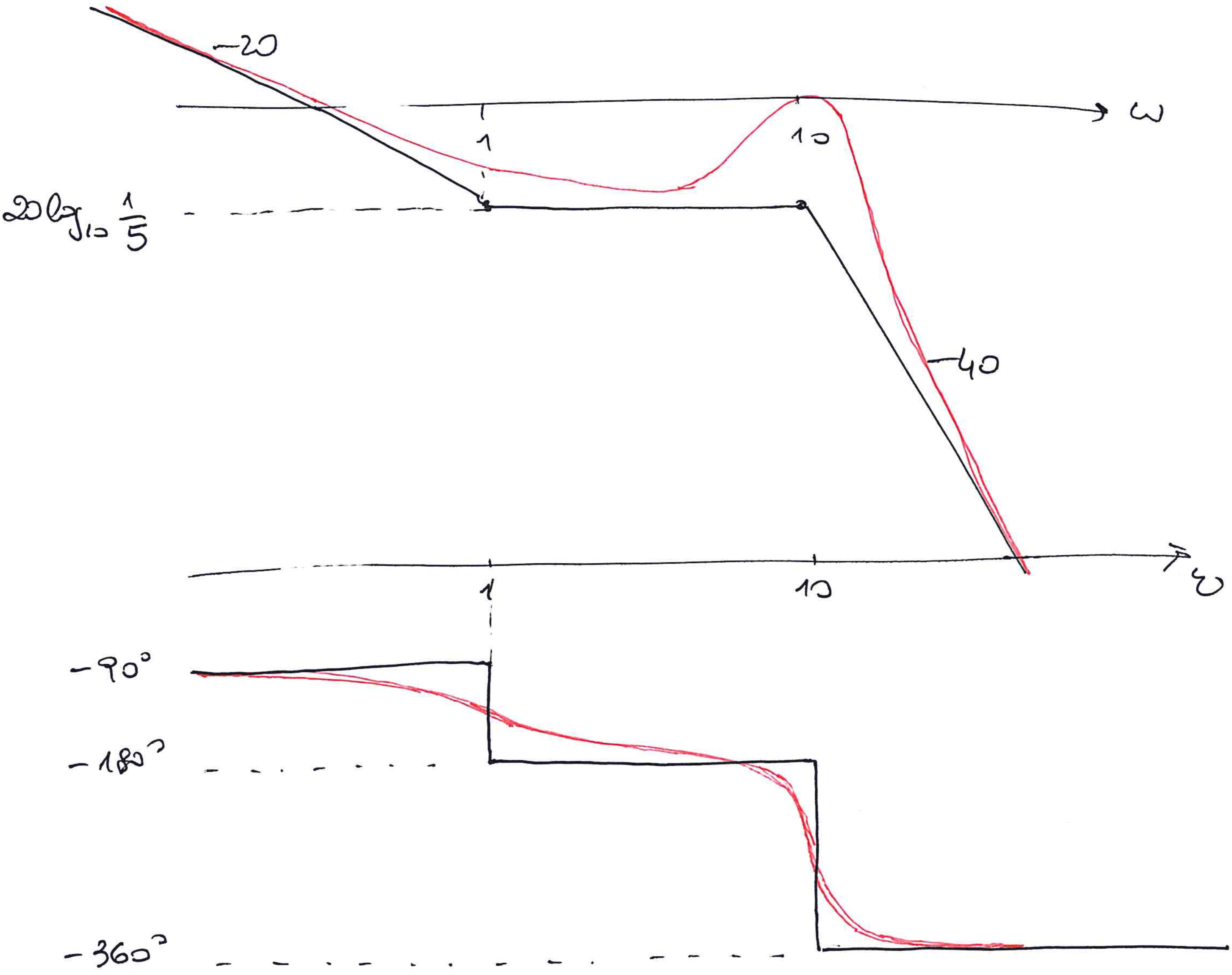
1. I

$$C(s)G(s) = \frac{1}{5}\ \frac{1-s}{s\left(1+\frac{2\cdot\frac{1}{10}}{10}s+\frac{1}{10^2}s^2\right)}$$



1. II

$$W(s) = \frac{C(s)G(s)}{1+\frac{1}{k}C(s)G(s)} = \frac{k\cdot\frac{1}{s}\cdot\frac{20(1-s)}{s^2+2s+100}}{k+\frac{1}{s}\frac{20(1-s)}{s^2+2s+100}} =$$

$$= \frac{20k(1-s)}{ks^3+2ks^2+(100k-20)s+20}$$

$$D(s) = \frac{G(s)}{1+\frac{1}{k}C(s)G(s)} = \frac{20ks(1-s)}{ks^3+2ks^2+(100k-20)s+20}$$

$$Q(s) = \frac{C(s)}{1+\frac{1}{k}C(s)G(s)} = \frac{k(s^2+2s+100)}{ks^3+2ks^2+(100k-20)s+20}$$

1. III Tabella di Routh

3	k	$100k-20$
2	$2k$	20
1	$\frac{200k^2-60k}{2k}$	
0	20	

$$\left.\begin{array}{l} k>0 \\ 100k-60>0 \end{array}\right\} \Rightarrow k > \frac{3}{10} = 0.3$$

1. IV Per l'esistenza del guadagno: $k>0.3$.

$$Q(0) = \frac{100k}{20} = 5k < 10 \Rightarrow k<2$$

$$\Rightarrow k \in (0.3, 2)$$

Poichè $W(0) = k$, si ha $Q(0) = 5\,W(0)$ e quindi il valore asintotico di $u(t)$ sarà sempre 5 volte quello di $y(t)$.

1. V Poichè $W(s)$ e $D(s)$ hanno lo stesso denominatore è sufficiente confrontare i numeratori di $|W(j\omega)|$ e $|D(j\omega)|$ ovvero:

$$|20k(1-j\omega)| \quad \text{e} \quad |20k\,j\omega(1-j\omega)| = \omega \cdot |20k(1-j\omega)|$$

Si può quindi concludere che per $\omega > 1$ è maggiore l'ampiezza della $y_{perm}(t)$ relativa all'ingresso $d(t)$, mentre per $\omega < 1$ quella dell'ingresso $r(t)$.

2.I
$$x(k) = \begin{bmatrix} y(k-2) \\ y(k-1) \\ y(k) \end{bmatrix} \qquad \begin{aligned} x_1(k+1) &= x_2(k) \\ x_2(k+1) &= x_3(k) \\ x_3(k+1) &= \tfrac{1}{2}x_3(k) + \tfrac{1}{3}x_2(k) + \tfrac{1}{6}x_1(k) + u(k) \\ y(k) &= x_3(k) \end{aligned}$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad C = [0 \quad 0 \quad 1] \quad D = [0]$$

2.II
$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -\frac{1}{6} & -\frac{1}{3} & \lambda - \frac{1}{2} \end{bmatrix} = \lambda^3 - \frac{1}{2}\lambda^2 - \frac{1}{3}\lambda - \frac{1}{6}$$

Autovalori: $\lambda_1 = 1 \qquad \lambda_{2,3} = -\frac{1}{4} \pm j\frac{1}{4}\sqrt{\frac{5}{3}}$

$\Rightarrow$ Funzione di trasferimento $G(z) = \dfrac{z^2}{(z-1)(z^2 + \frac{1}{2}z + \frac{1}{6})}$

$\Rightarrow$ Il sistema <u>non</u> è ILUL stabile.

2.III
$$X_\ell(z) = z\,C(zI-A)^{-1} \cdot x(0) = z \cdot [0 \quad 0 \quad 1] \begin{bmatrix} z & -1 & 0 \\ 0 & z & -1 \\ -\frac{1}{6} & -\frac{1}{3} & z - \frac{1}{2} \end{bmatrix}^{-1} x(0)$$

$$= z \cdot \frac{\left[\frac{1}{6}z \quad \frac{1}{3}z + \frac{1}{6} \quad z^2\right]}{z^3 - \frac{1}{2}z^2 - \frac{1}{3}z - \frac{1}{6}} \cdot x(0)$$

Poiché $X_\ell(z)$ ha un polo in $z = 1$, dal teorema del valore finale è sufficiente imporre:

$$\lim_{z \to 1} (z-1) X_\ell(z) = 0$$

$$\lim_{z \to 1} (z-1) z \cdot \frac{\left[\frac{1}{6}z \quad \frac{1}{3}z+\frac{1}{6} \quad z^2\right] \cdot x(0)}{(z-1)\left(z^2+\frac{1}{2}z+\frac{1}{6}\right)} = 0$$

$$\left[\frac{1}{6} \quad \frac{1}{2} \quad 1\right] x(0) = 0$$

Tutte le condizioni iniziali del tipo $\frac{1}{6}x_1(0)+\frac{1}{2}x_2(0)+ + x_3(0)=0$ hanno risposta libera convergente.

2.IV

$$Y_f(z) = \frac{z}{(z-1)^2} \qquad G(z) = \frac{z^2}{z^3-\frac{1}{2}z^2-\frac{1}{3}z-\frac{1}{6}}$$

$$U(z) = \frac{Y_f(z)}{G(z)} = \frac{z^2+\frac{1}{2}z+\frac{1}{6}}{z(z-1)} = 1 + \frac{\frac{3}{2}z+\frac{1}{6}}{z(z-1)} =$$

$$= 1 - \frac{\frac{1}{6}}{z} + \frac{\frac{5}{3}}{z-1}$$

$$\Rightarrow \quad u(k) = \delta(k) - \frac{1}{6}\delta(k-1) + \frac{5}{3} \cdot 1\!\!1(k-1)$$

2.V Dall'equazione i/o:

$$u(k-1) = -\frac{1}{2}y(k-1) - \frac{1}{3}y(k-2) - \frac{1}{6}y(k-3)$$

ovvero:

$$u(k) = -\frac{1}{2}y(k) - \frac{1}{3}y(k-1) - \frac{1}{6}y(k-2)$$

[oppure con la retroazione dello stato, allocando i poli tutti in 0].

3.I

$$\begin{cases} -\alpha x_1 - \beta x_1 x_2 + k = 0 \\ \beta x_1 x_2 - (\alpha+\gamma) x_2 = 0 \\ \gamma x_2 - \alpha x_3 = 0 \end{cases}$$

Dalla 2ª equazione: $x_2\left[\beta x_1 - (\alpha+\gamma)\right] = 0$

$$x_2 = 0 \Rightarrow x_3 = 0, \quad x_1 = \frac{k}{\alpha}$$

$$x_1 = \frac{\alpha+\gamma}{\beta} \Rightarrow x_2 = \frac{k - \alpha x_1}{\beta x_1} = \frac{k - \alpha \frac{\alpha+\gamma}{\beta}}{\alpha+\gamma} = \frac{\alpha}{\beta}\left(\frac{k\beta}{\alpha(\alpha+\gamma)} - 1\right)$$

$$x_3 = \frac{\gamma}{\alpha} x_2 = \frac{\gamma}{\beta}\left(\frac{k\beta}{\alpha(\alpha+\gamma)} - 1\right)$$

$$\bar{X}_A = \left(\frac{k}{\alpha}, 0, 0\right)^T$$

$$\bar{X}_B = \left(\frac{\alpha+\gamma}{\beta}, \frac{\alpha}{\beta}\left(\frac{k\beta}{\alpha(\alpha+\gamma)} - 1\right), \frac{\gamma}{\beta}\left(\frac{k\beta}{\alpha(\alpha+\gamma)} - 1\right)\right)^T$$

3.II

$$J = \frac{\partial f}{\partial x} = \begin{bmatrix} -\alpha - \beta x_2 & -\beta x_1 & 0 \\ \beta x_2 & \beta x_1 - \alpha - \gamma & 0 \\ 0 & \gamma & -\alpha \end{bmatrix}$$

$$J\big|_{x=\bar{X}_A} = \begin{bmatrix} -\alpha & -\beta \frac{k}{\alpha} & 0 \\ 0 & \beta \frac{k}{\alpha} - \alpha - \gamma & 0 \\ 0 & \gamma & -\alpha \end{bmatrix}$$

Autovalori: $-\alpha, -\alpha, \; k\frac{\beta}{\alpha} - \alpha - \gamma$

$$\bar{x}_A \begin{cases} \text{asintoticamente stabile se} & k < \dfrac{\alpha(\alpha+\gamma)}{\beta} \\ \text{instabile se} & k > \dfrac{\alpha(\alpha+\gamma)}{\beta} \end{cases}$$

$$J_{|x=\bar{x}_B} = \begin{bmatrix} -\alpha-\alpha\left(\dfrac{k\beta}{\alpha(\alpha+\gamma)}-1\right) & -(\alpha+\gamma) & 0 \\ \alpha\left(\dfrac{k\beta}{\alpha(\alpha+\gamma)}-1\right) & 0 & 0 \\ 0 & \gamma & -\alpha \end{bmatrix}$$

$$\det(\lambda I - J) = (\lambda+\alpha)\left[\lambda^2 + \alpha\frac{k\beta}{\alpha(\alpha+\gamma)}\lambda + \alpha(\alpha+\gamma)\left(\frac{k\beta}{\alpha(\alpha+\gamma)}-1\right)\right]$$

$$\bar{x}_B = \begin{cases} \text{asintoticamente stabile se} & k > \dfrac{\alpha(\alpha+\gamma)}{\beta} \\ \text{instabile se} & k < \dfrac{\alpha(\alpha+\gamma)}{\beta} \end{cases}$$

3.III

a) $\dot{x}_1(t) + \dot{x}_2(t) + \dot{x}_3(t) = -\beta x_1(t) x_2(t) + \beta x_1(t) x_2(t)$
$-\gamma x_2(t) + \gamma x_2(t) = 0$

b) $\dot{x}_1(t) = \dfrac{\partial}{\partial t}\left\{ x_1(0)\, e^{-\frac{\beta}{\gamma}(x_3(t)-x_3(0))} \right\} =$

$= -\dfrac{\beta}{\gamma} \cdot x_1(0)\, e^{-\frac{\beta}{\gamma}(x_3(t)-x_3(0))} \cdot \dot{x}_3(t) =$

$= -\dfrac{\beta}{\gamma} x_1(t)\, \dot{x}_3(t)$

Sostituendo $\dot{x}_1(t) = -\beta x_1(t) x_2(t)$ e $\dot{x}_3(t) = \gamma x_2(t)$ la relazione è verificata.

4.I

$$x_1(k+1) = x_1(k) - \alpha_{13} x_1(k) + \alpha_{21} x_2(k) + u(k)$$
$$x_2(k+1) = x_2(k) - \alpha_{21} x_2(k) - \alpha_{23} x_2(k) + \alpha_{42} x_4(k)$$
$$x_3(k+1) = x_3(k) - \alpha_{34} x_3(k) + \alpha_{13} x_1(k) + \alpha_{23} x_2(k)$$
$$x_4(k+1) = x_4(k) - \alpha_{42} x_4(k) + \alpha_{34} x_3(k)$$

$$y(k) = x_1(k)$$

$$A = \begin{bmatrix} 1-\alpha_{13} & \alpha_{21} & 0 & 0 \\ 0 & 1-\alpha_{21}-\alpha_{23} & 0 & \alpha_{42} \\ \alpha_{13} & \alpha_{23} & 1-\alpha_{34} & 0 \\ 0 & 0 & \alpha_{34} & 1-\alpha_{42} \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \quad D = [0]$$

4.II

$$R = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 1-\alpha_{13} & (1-\alpha_{13})^2 & (1-\alpha_{13})^3 \\ 0 & 0 & 0 & \alpha_{42}\alpha_{34}\alpha_{13} \\ 0 & \alpha_{13} & \alpha_{13}(1-\alpha_{13})+\alpha_{13}(1-\alpha_{34}) & * \\ 0 & 0 & \alpha_{34}\alpha_{13} & * \end{bmatrix}$$

$$\det R = \alpha_{13}^3 \alpha_{34}^2 \alpha_{42}$$

Il sistema è completamente raggiungibile sse

$\alpha_{13} \neq 0$, $\alpha_{34} \neq 0$, $\alpha_{42} \neq 0$

(altrimenti il grafo non è connesso e alcuni nodi non sono più "raggiungibili" nel vero senso della parola!)

4.III

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\det(\lambda I - A) = \lambda^4 - \frac{3}{2}\lambda^3 + \frac{3}{4}\lambda^2 - \frac{1}{4}\lambda = \lambda(\lambda-1)\left(\lambda^2 - \frac{1}{2}\lambda + \frac{1}{4}\right)$$

$$\lambda_1 = 0 \quad \lambda_2 = 1 \quad \lambda_{3,4} = \frac{1}{4} \pm j\frac{\sqrt{3}}{4} = \frac{1}{2}e^{\pm j\frac{\pi}{3}}$$

modi: $\delta(k)$, $\mathbb{1}(k)$, $\left(\frac{1}{2}\right)^k \cos\left(\frac{\pi}{3}k\right)$, $\left(\frac{1}{2}\right)^k \sin\left(\frac{\pi}{3}k\right)$

$$x_\ell(k) = A^k x(0)$$

$$\lim_{k\to+\infty} x_\ell(k) = \lim_{z\to 1} (z-1)z(zI-A)^{-1}x(0)$$

$$(zI-A)^{-1} = \begin{pmatrix} z-\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & z & 0 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & z-\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & z-\frac{1}{2} \end{pmatrix}^{-1} =$$

$$= \frac{\begin{pmatrix} z\left(z-\frac{1}{2}\right)^2-\frac{1}{8} & \frac{1}{2}\left(z-\frac{1}{2}\right)^2 & \frac{1}{8} & \frac{1}{4}\left(z-\frac{1}{2}\right) \\ \frac{1}{8} & \left(z-\frac{1}{2}\right)^3 & \frac{1}{4}\left(z-\frac{1}{2}\right) & \frac{1}{2}\left(z-\frac{1}{2}\right)^2 \\ \frac{1}{2}z\left(z-\frac{1}{2}\right) & \frac{1}{2}z\left(z-\frac{1}{2}\right) & z\left(z-\frac{1}{2}\right)^2 & \frac{1}{4}z \\ \frac{1}{4}z & \frac{1}{4}z & \frac{1}{2}z\left(z-\frac{1}{2}\right) & z\left(z-\frac{1}{2}\right)^2 \end{pmatrix}}{z(z-1)\left(z^2-\frac{1}{2}z+\frac{1}{4}\right)}$$

$$\lim_{z\to 1} (z-1)z(zI-A)^{-1}x(0) =$$

$$= \frac{1}{\frac{3}{4}} \begin{pmatrix} \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} x(0)$$

e quindi $\forall\, x(0)$ tale che $x_1(0)+x_2(0)+x_3(0)+x_4(0)=1$ si ha $\lim_{k\to+\infty} x_\ell(k) = \begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$.

4.IV $\quad \mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{4} \end{bmatrix}$

Il sistema è completamente osservabile.

Imponendo $\det(\lambda I - A + LC) = \lambda^4$ si ha

$$\det \begin{pmatrix} \lambda - \frac{1}{2} + l_1 & -\frac{1}{2} & 0 & 0 \\ l_2 & \lambda & 0 & -\frac{1}{2} \\ -\frac{1}{2} + l_3 & -\frac{1}{2} & \lambda - \frac{1}{2} & 0 \\ l_4 & 0 & -\frac{1}{2} & \lambda - \frac{1}{2} \end{pmatrix} =$$

$$= \left(\lambda - \frac{1}{2}\right)^2 \left(\lambda^2 + \left(l_1 - \frac{1}{2}\right)\lambda + \frac{1}{2} l_2\right) - \frac{1}{2} \cdot \left\{ \frac{1}{4}\left(\lambda - \frac{1}{2} + l_1\right) + \frac{1}{2}\left(\frac{1}{4} - \frac{1}{2} l_3 - l_4\left(\lambda - \frac{1}{2}\right)\right)\right\} =$$

$$= \lambda^4 + \left(l_1 - \frac{3}{2}\right)\lambda^3 + \left(\frac{1}{4} + \frac{1}{2} l_2 - l_1 + \frac{1}{2}\right)\lambda^2 + \left(\frac{1}{4} l_1 - \frac{1}{8} - \frac{1}{2} l_2 - \frac{1}{8} + \frac{1}{4} l_4\right)\lambda + \frac{1}{8} l_2 + \frac{1}{16} - \frac{1}{8} l_1 - \frac{1}{16} + \frac{1}{8} l_3 - \frac{1}{8} l_4 = \lambda^4$$

$$\Rightarrow \begin{cases} l_1 - \frac{3}{2} = 0 \\ \frac{1}{2} l_2 - l_1 + \frac{3}{4} = 0 \\ \frac{1}{4} l_1 - \frac{1}{2} l_2 + \frac{1}{4} l_4 - \frac{1}{4} = 0 \\ \frac{1}{8} l_2 - \frac{1}{8} l_1 + \frac{1}{8} l_3 - \frac{1}{8} l_4 = 0 \end{cases} \Rightarrow \begin{matrix} l_1 = \frac{3}{2} \\ l_2 = \frac{3}{2} \\ l_3 = \frac{5}{2} \\ l_4 = \frac{5}{2} \end{matrix}$$

$$L = \begin{pmatrix} \frac{3}{2} & \frac{3}{2} & \frac{5}{2} & \frac{5}{2} \end{pmatrix}^T$$

5.I Se $x_2(0)>0$ la seconda equazione è

$\dot{x}_2(t) = x_2(t)$ che ha come soluzione

$x_2(t) = x_2(0)\, e^{t} > 0 \quad \forall t$

e sostituendo nella prima:

$\dot{x}_1(t) = x_2(0)\, e^{t}$

da cui

$$x_1(t) = x_1(0) + \int_0^t x_2(0)\, e^{\tau}\, d\tau =$$

$$= x_1(0) + x_2(0)\,(e^{t} - 1)$$

Viceversa, se $x_2(0)<0$ si ha

$\dot{x}_2(t) = -x_2(t) \quad$ se $x_2(t)<0$

che ha come soluzione

$x_2(t) = x_2(0)\, e^{-t} < 0 \quad \forall t$!

e

$$x_1(t) = x_1(0) + \int_0^t x_2(0)\, e^{-\tau}\, d\tau =$$

$$= x_1(0) + x_2(0)\,(1 - e^{-t})$$

5.II Gli stati di equilibrio sono gli $\bar{x}$ del tipo $\bar{x} = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}$ $\alpha \in \mathbb{R}$.

Per ogni intorno di $\bar{x}$, $\exists\ x(0) = \begin{pmatrix} \alpha \\ \delta \end{pmatrix}$ con $\delta > 0$ tale per cui $x_2(t) = \delta e^{t} \to +\infty$