Analisi dei sistemi LTI a tempo continuo

Sia $t=0$ l'istante iniziale, tale che $u(t)=0 \ \forall t<0$.

Rappr. i/o

Dato il sistema

$$y^{(n)}(t)+a_{n-1}y^{(n-1)}(t)+\ldots+a_1\dot{y}(t)+a_0y(t)=$$
$$=b_m u^{(m)}(t)+b_{m-1}u^{(m-1)}(t)+\ldots+b_1\dot{u}(t)+b_0u(t)$$

determinare $y(t)\ \forall t>0$ a partire da $u(t)$ e dalle condizioni iniziali $y(0), \dot{y}(0), \ldots, y^{(n-1)}(0)$.

Rappr. i/s/o

Dato il sistema

$$\dot{x}(t)=Ax(t)+Bu(t)$$
$$y(t)=Cx(t)+Du(t)$$

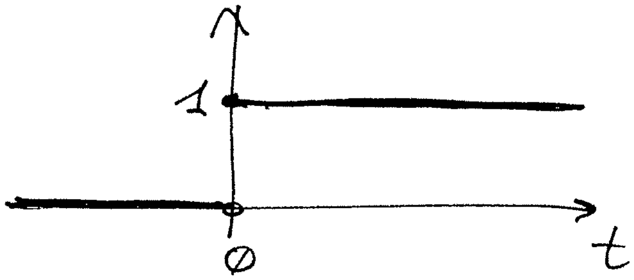
determinare $x(t)$ e $y(t)\ \forall t>0$, a partire da $u(t)$ e da $x(0)$.

$x(t)$: risposta totale nello stato
$y(t)$: " " nell' uscita

Segnali tipici nei sistemi a tempo-continuo

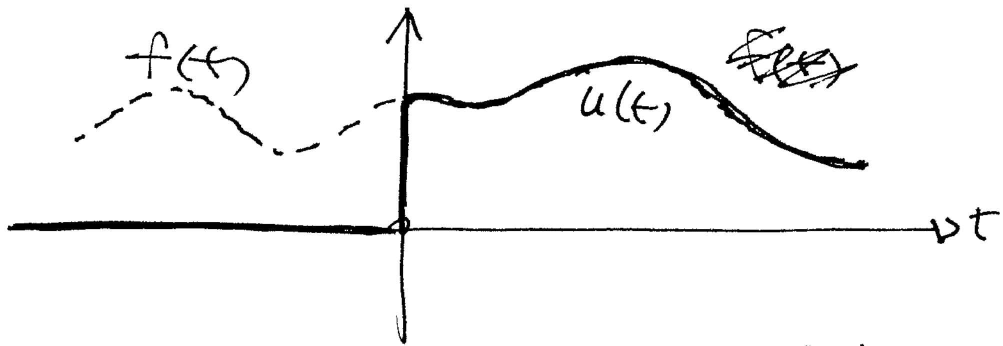
1) GRADINO UNITARIO

$$\mathbb{1}(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$



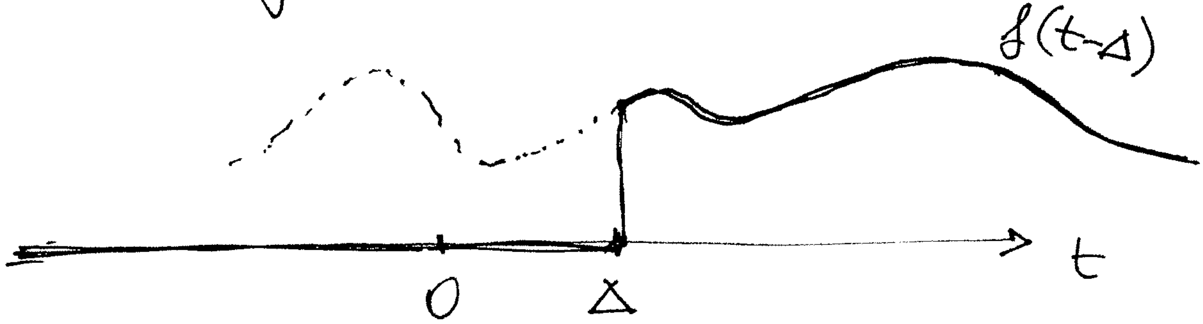
Tutti i segnali che consideriamo sono del tipo:

$$u(t) = f(t) \cdot \mathbb{1}(t) \qquad f : \mathbb{R} \to \mathbb{R}$$

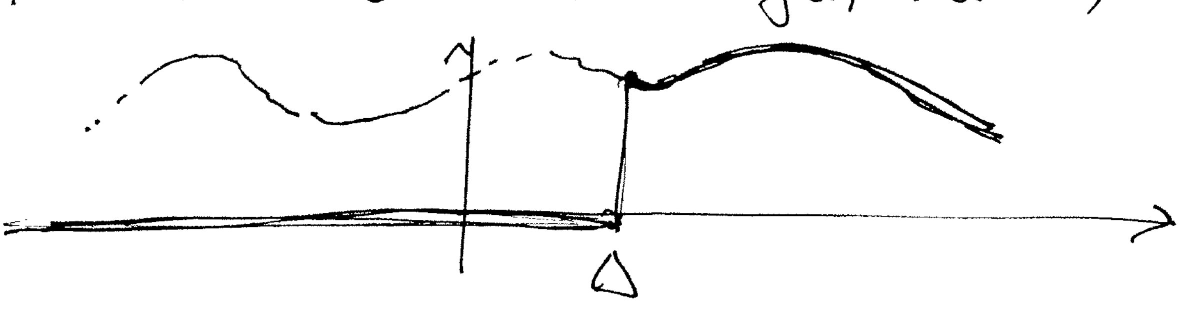


Operatore di traslazione nel tempo

$$u(t) = f(t-\Delta) \cdot \mathbb{1}(t-\Delta)$$



[Attenzione: è diverso da $f(t) \cdot \mathbb{1}(t-\Delta)$

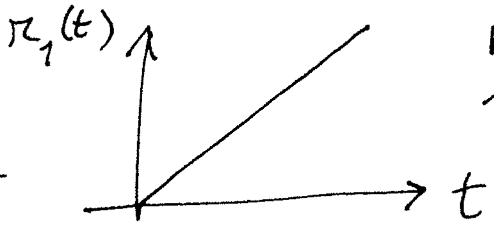


2) Segnali canonici

$$r_k(t) = \frac{t^k}{k!} \cdot \mathbb{1}(t) \qquad k = 0, 1, 2, \dots$$

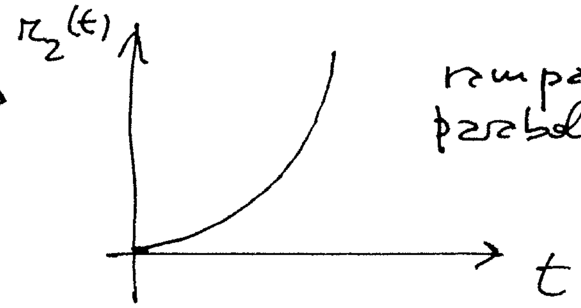
$$r_0(t) = \mathbb{1}(t)$$

$$r_1(t) = t \cdot \mathbb{1}(t) \rightarrow$$



rampa lineare

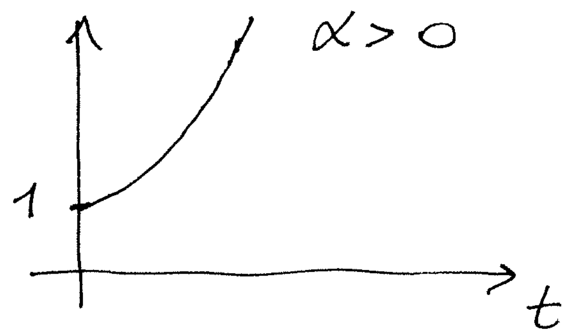
$$r_2(t) = \frac{t^2}{2} \cdot \mathbb{1}(t) \rightarrow$$



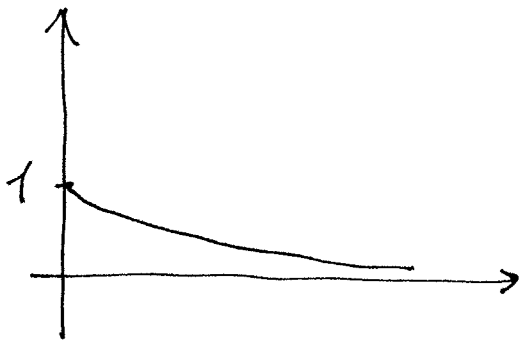
rampa parabolica

3) Segnali esponenziali

$$f(t) = e^{\alpha t} \cdot \mathbb{1}(t) \qquad \alpha \in \mathbb{R}$$



$\alpha > 0$



$\alpha < 0$

$$\alpha = 0 : \quad f(t) = \mathbb{1}(t)$$

4) Segnali a rampa esponenziale

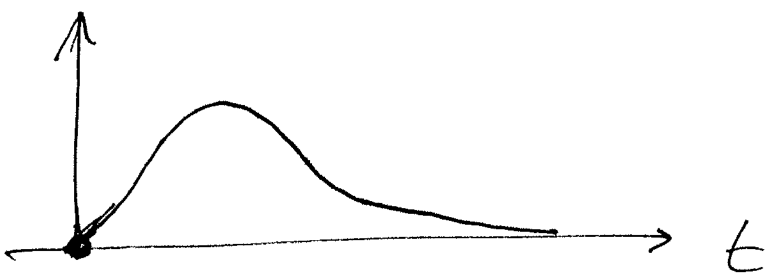
$$f_k(t) = \frac{t^k}{k!} e^{\alpha t} \cdot \mathbb{1}(t)$$

$k=0 \rightarrow$ segnale esponenziale

$\alpha=0 \rightarrow$ segnali canonici

<u>Es</u>. $k=1, \alpha<0$

$$f_1(t) = t e^{\alpha t} \cdot \mathbb{1}(t) \quad , \alpha < 0$$



5) Segnali sinusoidali

$$f(t) = \cos(\omega t) \cdot \mathbb{1}(t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2} \cdot \mathbb{1}(t) =$$

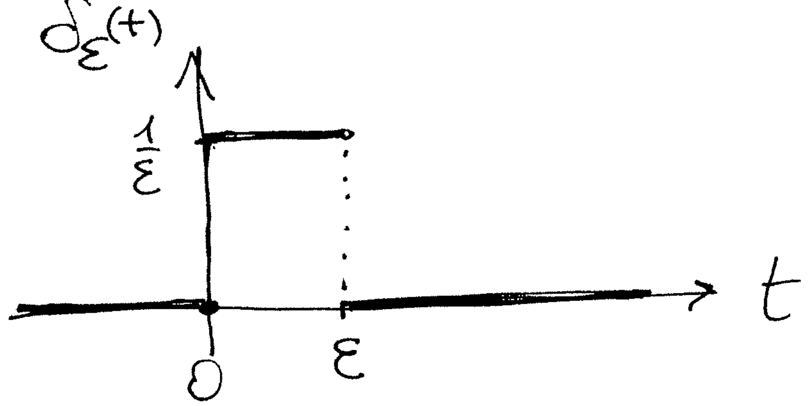
$$= \frac{1}{2} \cdot e^{j\omega t} \cdot \mathbb{1}(t) + \frac{1}{2} e^{-j\omega t} \mathbb{1}(t)$$

$$f(t) = \sin(\omega t) \cdot \mathbb{1}(t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j} \mathbb{1}(t)$$

6) Segnale impulsivo (delta di Dirac)

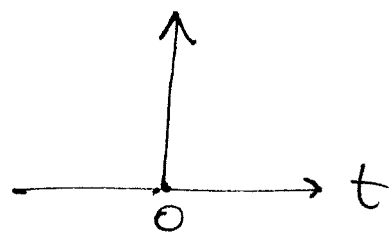
Definiamo il segnale

$$\delta_\varepsilon(t) = \begin{cases} \frac{1}{\varepsilon} & \text{se } 0 \leq t \leq \varepsilon \\ 0 & \text{altrimenti} \end{cases}$$



Si definisce DELTA DI DIRAC

$$\delta(t) = \lim_{\varepsilon \to 0} \delta_\varepsilon(t)$$



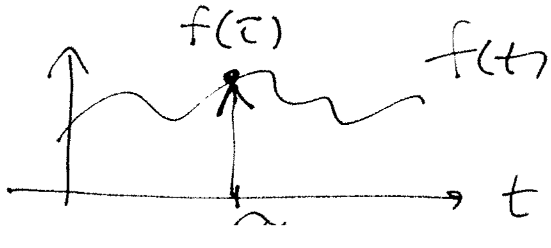
Proprietà

a) $$\int_{-\infty}^{+\infty} \delta(t)\, dt = 1$$

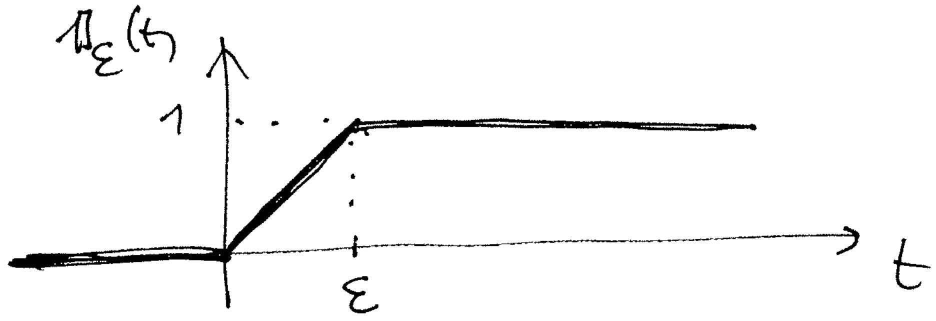
Deriva da $$\int_{-\infty}^{+\infty} \delta_\varepsilon(t)\, dt = 1 \quad \forall \varepsilon$$

b) Se $f(t)$ è una funzione continua, allora

$$\int_{-\infty}^{+\infty} f(t) \cdot \delta(t-\tau)\, dt = f(\tau)$$

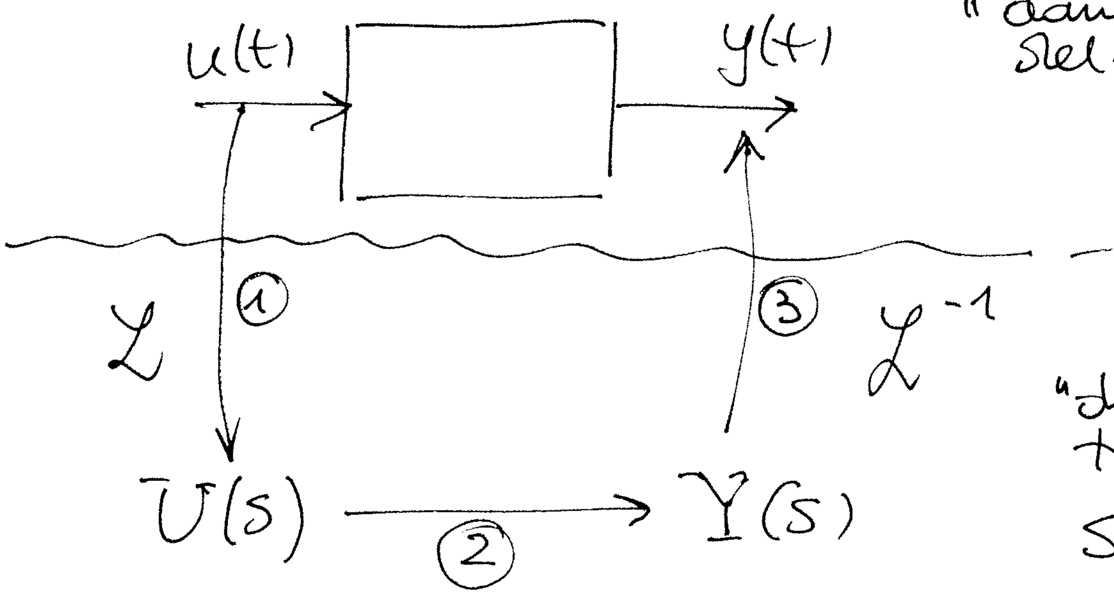


c) La funzione $\delta_\varepsilon(t)$ è la derivata della funzione $\mathbb{1}_\varepsilon(t)$:



Per estensione ($\varepsilon \to 0$), si considera $\delta(t)$ come la "derivata" di $\mathbb{1}(t)$ (e viceversa $\mathbb{1}(t)$ è l'"integrale" di $\delta(t)$)

TRASFORMATA DI LAPLACE



"dominio del tempo"

"dominio trasformato"

$s \in \mathbb{C}$

<u>Definizione</u>.

Sia $f(t): \mathbb{R} \to \mathbb{R}$, tale che $f(t) = 0 \ \forall t < 0$, $\exists M > 0$, $t_0 \geq 0$ ed $\alpha \in \mathbb{R}$: $|f(t)| \leq M e^{\alpha t}$, $\forall t \geq t_0$. Si definisce allora <u>trasformata di Laplace</u> di $f(t)$ la funzione $F: \mathbb{C} \to \mathbb{C}$

$$F(s) = \int_0^{+\infty} f(t)\, e^{-st}\, dt .$$

e si indica con $F(s) = \mathcal{L}[f(t)]$

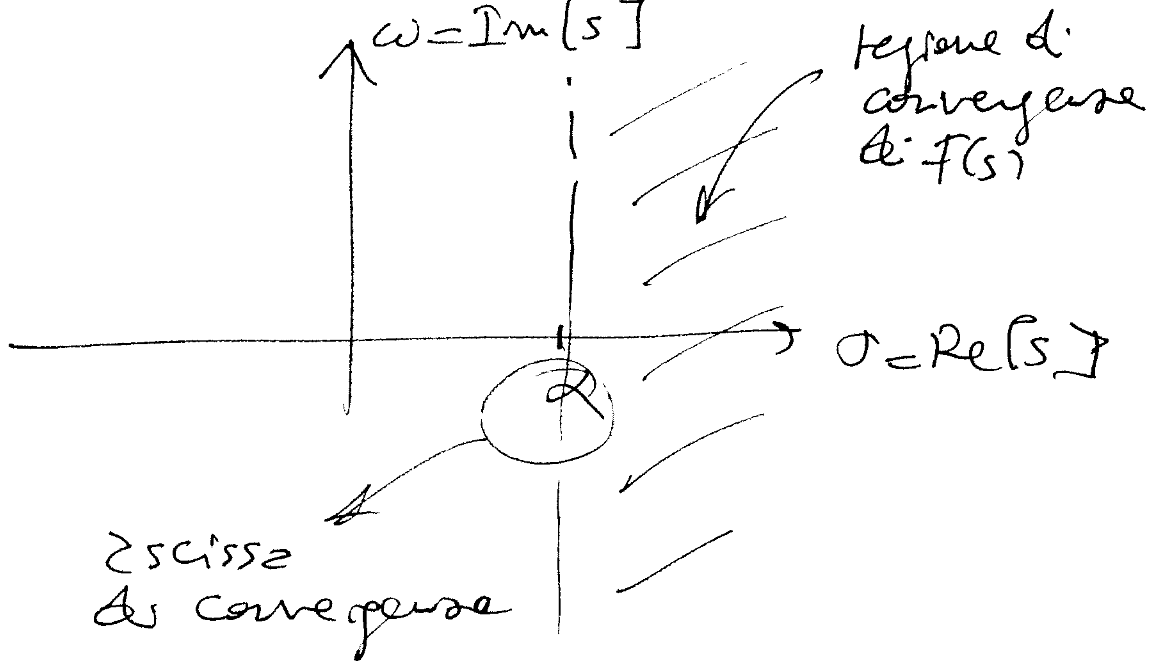
L'integrale di $F(s)$ converge?

$$|f(t)e^{-st}| \underset{\substack{\uparrow \\ s=\sigma+j\omega}}{=} |f(t)\,e^{-\sigma t}e^{-j\omega t}| =$$

$$= |f(t)|\cdot e^{-\sigma t}\cdot \underbrace{|e^{-j\omega t}|}_{1} \le$$

$$\le M\cdot e^{\alpha t}e^{-\sigma t} = M e^{(\alpha-\sigma)t}$$

La $F(s)$ converge $\forall\, \sigma > \alpha$



Proprietà della trasformata di Laplace

1) Linearità

$$f(t) = \alpha_1 f_1(t) + \alpha_2 f_2(t) \qquad \alpha_1, \alpha_2 \in \mathbb{R}$$

$$F(s) = \alpha_1 F_1(s) + \alpha_2 F_2(s)$$

2) Trasformata di segnali con ritardo

$$g(t) = f(t-\Delta) \cdot \mathbb{1}(t-\Delta) \qquad \Delta \in \mathbb{R},\ \Delta > 0$$

$$G(s) = F(s)\, e^{-\Delta \cdot s}$$

Dim.

$$G(s) = \int_0^{+\infty} g(t)\, e^{-st}\, dt \underset{t = \tau + \Delta}{=} \int_{-\Delta}^{+\infty} g(\tau+\Delta)\, e^{-s(\tau+\Delta)}\, d\tau$$

$$= \int_{-\Delta}^{+\infty} f(\tau) \cdot \mathbb{1}(\tau)\, e^{-s\tau} \cdot e^{-s\Delta}\, d\tau =$$

$$= e^{-s\Delta} \cdot \underbrace{\int_0^{+\infty} f(\tau) \cdot \mathbb{1}(\tau)\, e^{-s\tau}\, d\tau}_{F(s)}$$

3) Moltiplicazione per un esponenziale

$$g(t) = f(t)\, e^{\gamma t} \qquad \gamma \in \mathbb{R}$$

$$G(s) = F(s-\gamma)$$

Dim.

$$G(s) = \int_0^{+\infty} \underbrace{f(t)\, e^{\gamma t}}_{g(t)}\, e^{-st}\, dt =$$

$$= \int_0^{+\infty} f(t)\, e^{-(s-\gamma)t}\, dt = F(s-\gamma)$$

Alcune trasformate di segnali notevoli

* $\mathbb{1}(t)$

$$\int_0^{+\infty} \mathbb{1}(t)\, e^{-st}\, dt = \frac{1}{(-s)} e^{-st} \Big|_0^{+\infty} =$$

$$= 0 - \frac{1}{(-s)} e^{0} = \frac{1}{s}$$

$$\mathcal{L}[\mathbb{1}(t)] = \frac{1}{s}$$

* $\delta(t)$

$$\int_0^{+\infty} \delta(t)\, \underbrace{e^{-st}}_{f(t)}\, dt = e^{-st}\Big|_{t=0} = 1$$

$$\mathcal{L}[\delta(t)] = 1$$

* segnale esponenziale $e^{\alpha t} \cdot \mathbb{1}(t)$

$$g(t) = \underbrace{\mathbb{1}(t)}_{f(t)} \cdot e^{\alpha t}$$

$$G(s) = F(s-\alpha) = \frac{1}{s-\alpha}$$

$$\mathcal{L}\left[e^{\alpha t} \cdot \mathbb{1}(t)\right] = \frac{1}{s-\alpha}$$

4) Moltiplicazione per t

$$g(t) = t \cdot f(t)$$

$$G(s) = -\frac{d}{ds} F(s)$$

Dim.

$$\frac{d}{ds} F(s) = \frac{d}{ds} \int_0^{+\infty} f(t) e^{-st} dt =$$

$$= \int_0^{+\infty} f(t) \left[\frac{d}{ds} e^{-st}\right] dt =$$

$$= \int_0^{+\infty} f(t) (-t) e^{-st} dt =$$

$$= -\int_0^{+\infty} [t \cdot f(t)] e^{-st} dt$$

* trasformata dei segnali canonici

$$r_k(t) = \frac{t^k}{k!}\,\mathbb{1}(t)$$

$$r_1(t) = t \cdot \underbrace{\mathbb{1}(t)}_{f(t)}$$

$$\mathcal{L}[r_1(t)] = R_1(s) = -\frac{d}{ds}\frac{1}{s} = \frac{1}{s^2}$$

$$r_2(t) = \frac{t}{2}\cdot r_1(t) = \frac{t^2}{2}\cdot \mathbb{1}(t)$$

$$\mathcal{L}[r_2(t)] = R_2(s) = \frac{1}{2}\cdot\left(-\frac{d}{ds}R_1(s)\right) =$$

$$= -\frac{1}{2}\frac{d}{ds}\frac{1}{s^2} = -\frac{1}{2}\left(\frac{-2s}{s^4}\right) = \frac{1}{s^3}$$

$\vdots$

$$\mathcal{L}[r_k(t)] = \frac{1}{s^{k+1}}$$

* rampa esponenziale

$$f_k(t) = \underbrace{\frac{t^k}{k!}}_{\downarrow \mathcal{L} \; \frac{1}{s^{k+1}}} e^{\alpha t} \cdot 1(t)$$

$$\mathcal{L}\left[f_k(t)\right] = F_k(s) = \frac{1}{(s-\alpha)^{k+1}}$$

* segnali sinusoidali

$$u(t) = \cos(\omega t) \cdot 1(t)$$

$$U(s) = \mathcal{L}\left[\frac{e^{j\omega t} + e^{-j\omega t}}{2} 1(t)\right] =$$

$$= \frac{1}{2} \cdot \mathcal{L}\left[e^{\underline{j\omega} t} \cdot 1(t)\right] + \frac{1}{2} \mathcal{L}\left[e^{\underline{-j\omega} t} \cdot 1(t)\right] =$$

$$= \frac{1}{2} \cdot \frac{1}{s - j\omega} + \frac{1}{2} \cdot \frac{1}{s + j\omega} =$$

$$= \frac{s + \cancel{j\omega} + s - \cancel{j\omega}}{2(s-j\omega)(s+j\omega)} = \frac{s}{s^2 + \omega^2}$$

$$\mathcal{L}\left[\sin(\omega t) \cdot 1(t)\right] = \frac{\omega}{s^2 + \omega^2}$$

5) Trasformata della derivata

$$g(t) = \dot{f}(t) \cdot \mathbb{1}(t) \qquad \mathcal{L}[f(t)] = F(s)$$

$$G(s) = \mathcal{L}[\dot{f}(t)] = \underline{s \cdot F(s) - f(\emptyset)}$$

Dim.

per parti

$$G(s) = \int_0^{+\infty} \dot{f}(t)\, e^{-st}\, dt =$$

$$= f(t) e^{-st} \Big|_{\emptyset}^{+\infty} - \int_0^{+\infty} f(t)(-s) e^{-st}\, dt =$$

$$= \emptyset - f(0) + s \underbrace{\int_0^{+\infty} f(t) e^{-st}\, dt}_{F(s)}$$

$$|f(t) e^{-st}| \leq M e^{-(\sigma-\alpha)t} \xrightarrow[t \to +\infty]{} 0 \quad \forall \sigma > \alpha$$

6) Trasformata dell'integrale

$$g(t) = \int_0^t f(\tau)\, d\tau$$

$$G(s) = \mathcal{L}\left[\int_0^t f(\tau)\, d\tau\right] = \frac{1}{s} F(s)$$

Dim.

$$\dot{g}(t) = f(t)$$

$$\mathcal{L}[\dot{g}(t)] = s G(s) - \cancel{g(\emptyset)}^{\,\emptyset} = F(s)$$

7) Teorema del valore finale

Se $\lim_{t \to +\infty} f(t)$ esiste finito, allora

$$\lim_{t \to +\infty} f(t) = \lim_{s \to 0} s \cdot F(s)$$

<u>Esempi</u>

* $f(t) = e^{-t}$

$$\lim_{t \to +\infty} e^{-t} = \emptyset = \lim_{s \to 0} s \cdot \frac{1}{s+1} = \emptyset$$

* $f(t) = \mathbb{1}(t)$

$$\lim_{t \to \infty} \mathbb{1}(t) = 1 = \lim_{s \to 0} s \cdot \frac{1}{s} = 1$$

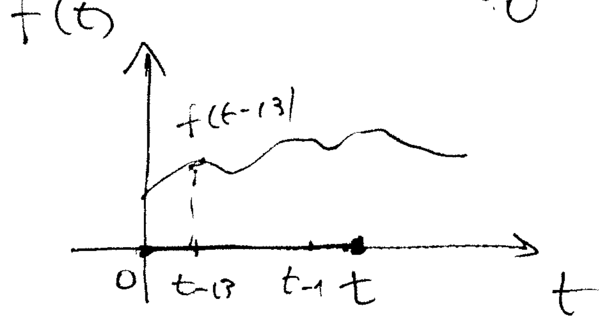
* $f(t) = e^{t}$

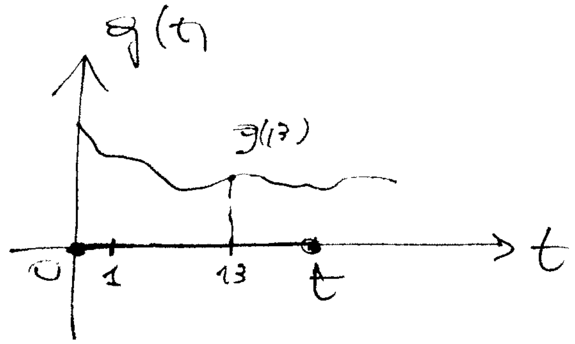
$$\lim_{t \to +\infty} e^{t} = +\infty \qquad \lim_{s \to 0} s \cdot \frac{1}{s-1} = \emptyset$$

8) Trasformata del prodotto di convoluzione

Si definisce prodotto di convoluzione di due funzioni $f(t)$ e $g(t)$

$$p(t) = f(t) * g(t) = \int_0^t f(t-\tau)\, g(\tau)\, d\tau$$



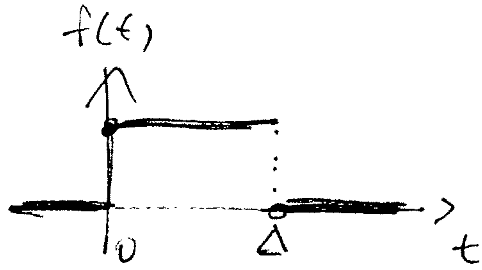


$$P(s) = \mathcal{L}\left[f(t) * g(t) \right] = F(s) \cdot G(s)$$

Esempi di trasformate

(a) Funzione rettangolare

$$f(t) = \begin{cases} 1 & 0 \le t \le \Delta \\ 0 & \text{altrimenti} \end{cases}$$

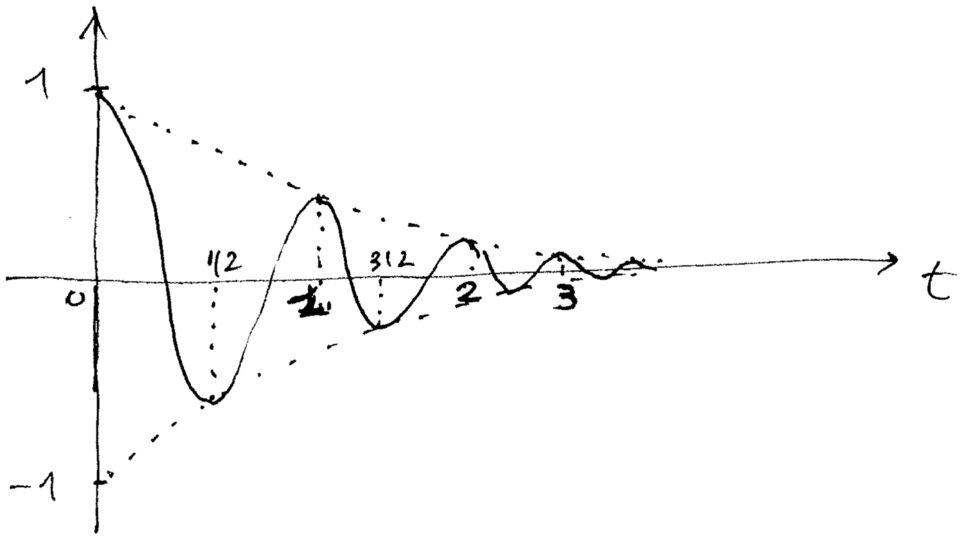


$$f(t) = 1\!\!1(t) - 1\!\!1(t-\Delta)$$

$$F(s) = \frac{1}{s} - \frac{1}{s} e^{-\Delta s} = \frac{1}{s}\left(1 - e^{-\Delta s}\right)$$

(b) Sinusoide smorzata

$$u(t) = e^{-2t} \cos(2\pi t) \cdot 1\!\!1(t)$$



$$U(s) = \mathcal{L}\left[\cos(2\pi t)\right]\Big|_{s \to s+2} = \frac{s}{s^2 + (2\pi)^2}\Big|_{s \to s+2} =$$

$$= \frac{s+2}{(s+2)^2 + 4\pi^2}$$

radici
$-2 \pm j\,2\pi$

(c) ... funzione incasinata!

$$f(t) = \int_0^t \tau\, e^{-\tau} \cos(2\tau)\, d\tau \cdot \mathbb{1}(t)$$

$$\mathcal{L}[\cos(2t)] = \frac{s}{s^2+4}$$

$$\mathcal{L}\left[e^{-t} \cos(2t)\right] = \frac{s+1}{(s+1)^2+4}$$

$$\mathcal{L}\left[t\, e^{-t} \cos(2t)\right] = -\frac{d}{ds}\, \frac{s+1}{(s+1)^2+4}$$

$$\mathcal{L}[f(t)] = \frac{1}{s}\left[-\frac{d}{ds}\left(\frac{s+1}{(s+1)^2+4}\right)\right]$$

Uso della trasformata di Laplace per la soluzione di sistemi LTI

Esempio: massa - molla - smorzatore

Equazione i/o: $M\ddot{y}(t) + \beta\dot{y}(t) + Ky(t) = u(t)$

$y \to$ posizione della massa
$u \to$ forza applicata alla massa

Trasformata dell'equazione

$$\mathcal{L}\left[M\ddot{y}(t) + \beta\dot{y}(t) + Ky(t)\right] = \mathcal{L}\left[u(t)\right] = U(s)$$

$$M \cdot \mathcal{L}\left[\ddot{y}(t)\right] + \beta\underbrace{\mathcal{L}\left[\dot{y}(t)\right]}_{sY(s)-y(0)} + K\underbrace{\mathcal{L}\left[y(t)\right]}_{Y(s)} = U(s)$$

$$\mathcal{L}\left[\ddot{y}(t)\right] = \mathcal{L}\left[\frac{d}{dt}\dot{y}(t)\right] = s \cdot \mathcal{L}\left[\dot{y}(t)\right] - \dot{y}(0) =$$
$$= s\left\{sY(s) - y(0)\right\} - \dot{y}(0) = s^2Y(s) - sy(0) - \dot{y}(0)$$

Sostituendo

$$M\left\{s^2Y(s) - sy(0) - \dot{y}(0)\right\} + \beta\left\{sY(s) - y(0)\right\}$$
$$+ KY(s) = U(s)$$

$$\left[Ms^2 + \beta s + K\right]Y(s) = Msy(0) + M\dot{y}(0) + \beta y(0)$$
$$+ U(s)$$

$$Y(s) = \underbrace{\frac{(My(0))s + (M\dot{y}(0) + \beta y(0))}{Ms^2 + \beta s + k}}_{Y_l(s)\ :\ \text{trasformata della risposta libera}} +$$

$$+ \underbrace{\frac{1}{Ms^2 + \beta s + k}}_{Y_f(s)\ :\ \text{trasformata della risposta forzata}} \cdot U(s)$$

$$Y_f(s) = G(s) \cdot U(s)$$

con $G(s) = \dfrac{1}{Ms^2 + \beta s + k}$ $\leftarrow$ funzione di trasferimento del sistema

$$y^{(n)}(t) = \frac{d^n}{dt^n} y(t)$$

$$\mathcal{L}\left[y^{(n)}(t) \right] = s^n Y(s) - s^{n-1} y(0) - s^{n-2} \dot{y}(0) - \dots$$

$$\dots - s\, y^{(n-2)}(0) - y^{(n-1)}(0)$$

Trasformata della rappr. i/o

$$\{\underline{s^n Y(s)} - s^{n-1} y(0) - s^{n-2}\dot{y}(0) - \dots - y^{(n-1)}(0)\} +$$

$$+ a_{n-1}\{\underline{s^{n-1} Y(s)} - s^{n-2} y(0) - s^{n-3}\dot{y}(0) - \dots - y^{(n-2)}(0)\} +$$

$$+ \dots + a_1 \{\underline{s Y(s)} - y(0)\} + a_0 Y(s) =$$

$$= b_m s^m \cdot U(s) + b_{m-1} s^{m-1} U(s) + \dots + b_1 s U(s) + b_0 U(s)$$

$$\{s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \dots + a_1 s + a_0\} Y(s) =$$

$$= \Big[s^{n-1} y(0) + s^{n-2}\dot{y}(0) + \dots + y^{(n-1)}(0) +$$

$$+ a_{n-1}\{s^{n-2} y(0) + s^{n-3}\dot{y}(0) + \dots + y^{(n-2)}(0)\} +$$

$$+ \dots + a_1 y(0) \Big] +$$

$$+ \{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0\} U(s)$$

Divido per $\{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0\}$ e ottengo

$$Y(s) = \underbrace{\frac{c_{n-1}s^{n-1} + c_{n-2}s^{n-2} + \dots + c_1 s + c_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0}}_{Y_l(s)} +$$

$$+ \underbrace{\frac{b_m s^m + b_{m-1}s^{m-1} + \dots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0} \cdot U(s)}_{Y_f(s)}$$

dove i coefficienti c_i, $i = 0, 1, \dots, n-1$ dipendono dagli a_i e dalle condizioni iniziali $y(0), \dot{y}(0), \dots, y^{(n-1)}(0)$.

$$Y_f(s) = G(s) \cdot U(s)$$

dove

$$G(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \dots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0}$$

è la FUNZIONE DI TRASFERIMENTO del sistema

OSSERVAZIONI

* $Y_\ell(s)$ è una funzione razionale fratta strettamente propria [grado denomin. > grado numer.]

* $Y_\ell(s)$ e $G(s)$ hanno lo stesso denominatore. Le radici del denominatore si chiamano "POLI" del sistema.

* Poiché $m \leq n$ (per la causalità), la funzione di trasferimento $G(s)$ è una funzione razionale fratta propria (se $m=n$) oppure strettamente propria (se $m<n$).

* La funzione di trasferimento $G(s)$ contiene tutta l'informazione della rappresentazione I/O

Trasformata di Laplace della rappresentazione i/s/u

$$\mathcal{L}[\dot{x}(t)] = \mathcal{L}[Ax(t) + Bu(t)]$$

$$s \cdot X(s) - x(0) = A \cdot X(s) + B \cdot U(s)$$

$$\underbrace{s}_{1\times 1}\underbrace{X(s)}_{n\times 1} - \underbrace{A}_{n\times n} \cdot \underbrace{X(s)}_{n\times 1} = x(0) + B \cdot U(s)$$

$$s \cdot I_{n\times n} \cdot X(s) - AX(s) = x(0) + B \cdot U(s)$$

$$(sI - A) \cdot X(s) = x(0) + B \cdot U(s)$$

$$X(s) = \underbrace{(sI-A)^{-1} x(0)}_{\substack{X_l(s) \\ \text{risposta libera} \\ \text{nello stato}}} + \underbrace{(sI-A)^{-1} B \cdot U(s)}_{\substack{X_f(s) \\ \text{risposta forzata} \\ \text{nello stato}}}$$

$$Y(s) = C X(s) + D U(s) =$$
$$= C\{(sI-A)^{-1}x(0) + (sI-A)^{-1}B \cdot U(s)\} + DU(s)$$
$$= C(sI-A)^{-1}x(0) + C(sI-A)^{-1}BU(s) + DU(s)$$
$$= \underbrace{C(sI-A)^{-1}x(0)}_{\substack{Y_l(s) \\ \text{risposta libera} \\ \text{nell'uscita}}} + \underbrace{\{C(sI-A)^{-1}B + D\}U(s)}_{\substack{Y_f(s) \\ \text{risposta forzata} \\ \text{nell'uscita}}}$$

$$Y_f(s) = \underbrace{\left\{ C(sI-A)^{-1}B + D \right\}}_{G(s)} \cdot U(s)$$

la funzione di trasferimento!

ANTITRASFORMATA DI LAPLACE

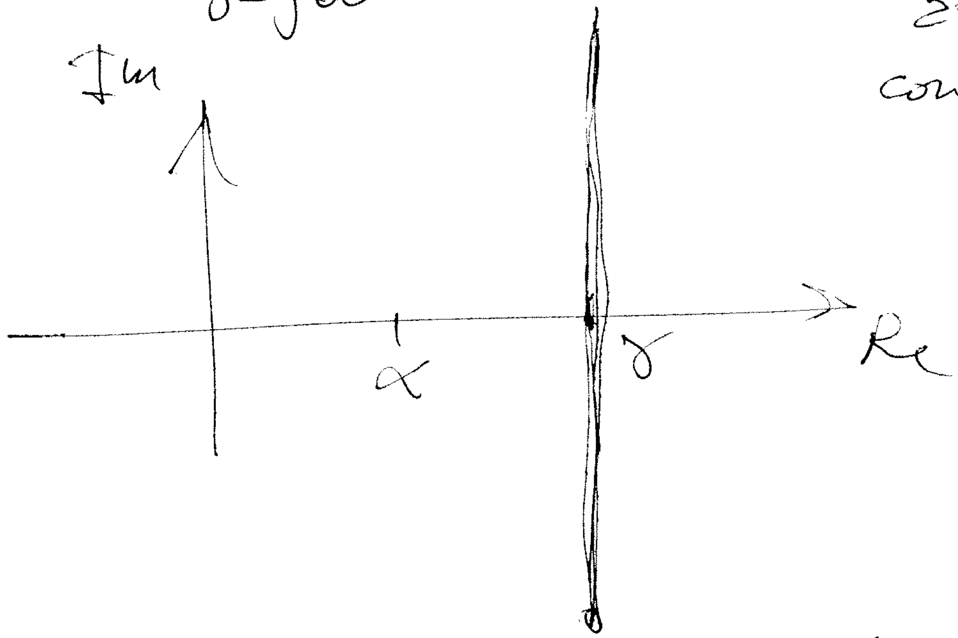
Problema: data $Y(s)$, determinare

$$y(t) = \mathcal{L}^{-1}[Y(s)]$$

Risultato:

$$y(t) = \frac{1}{2\pi j} \int_{\gamma - j\infty}^{\gamma + j\infty} Y(s)\, e^{st}\, ds \qquad \gamma > \alpha$$

ascissa di convergenza



A noi interessano le $Y(s)$ razionali fratte proprie e strettamente proprie

Esempio : mms

$$Y_\ell(s) = \frac{My(0)s + M\dot{y}(0) + \beta y(0)}{Ms^2 + \beta s + k}$$

$$Y_f(s) = \frac{1}{Ms^2 + \beta s + k} \cdot U(s)$$

Calcolare $y_\ell(t)$ per $M = 1$, $\beta = 3$, $k = 2$,

$y(0) = 1$, $\dot{y}(0) = 0$.

$$Y_\ell(s) = \frac{s + 3}{s^2 + 3s + 2}$$

1° step : calcolo le radici del denominatore (i poli del sistema)

$$s = \frac{-3 \pm \sqrt{9 - 8}}{2} = \begin{cases} -1 \\ -2 \end{cases}$$

$$Y_\ell(s) = \frac{s+3}{(s+1)(s+2)}$$

2° step : scomposizione in fratti semplici

$$Y_\ell(s) = \frac{R_1}{s+1} + \frac{R_2}{s+2}$$

sviluppo di Heaviside

$$= \frac{R_1(s+2) + R_2(s+1)}{(s+1)(s+2)}$$

$$R_1(s+2) + R_2(s+1) = s+3$$

$$(R_1+R_2)s + 2R_1 + R_2 = 1 \cdot s + 3$$

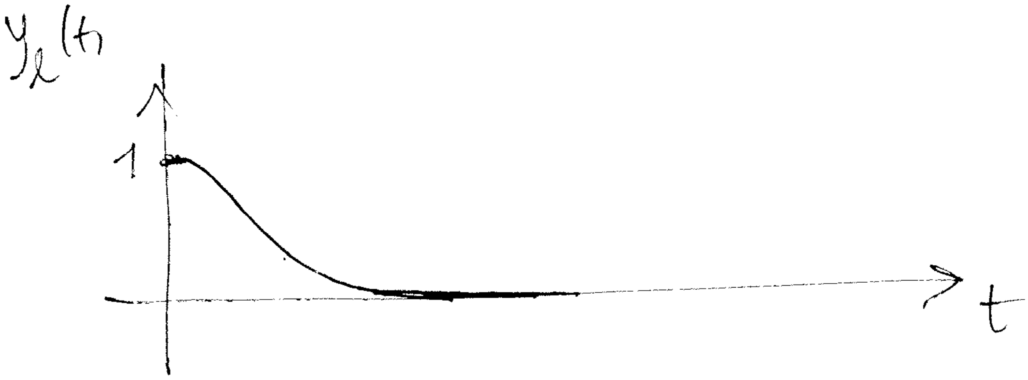
$$\begin{cases} R_1 + R_2 = 1 \\ 2R_1 + R_2 = 3 \end{cases} \qquad \begin{aligned} R_1 &= 2 \\ R_2 &= -1 \end{aligned}$$

$$Y_\ell(s) = \frac{2}{s+1} - \frac{1}{s+2}$$

3° step: antitrasformata

$$y_\ell(t) = 2 \cdot \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+2}\right] =$$

$$= \left\{2e^{-t} - e^{-2t}\right\} \cdot 1(t)$$



Antitrasformata di funzioni razionali fratte

$$F(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0}$$

$n \geq m$

<u>Caso 1</u>. $F(s)$ strettamente propria ($n > m$) e poli semplici (le radici di $D(s)$ hanno molteplicità algebrica 1)

Posso scrivere $F(s)$ nella <u>forma zeri-poli</u>

$$F(s) = K \frac{(s-z_1)(s-z_2)\dots(s-z_m)}{(s-p_1)(s-p_2)\dots(s-p_n)}$$

z_j : zeri $\quad$ p_j : poli $\quad$ $z_j, p_j \in \mathbb{C}$

poli semplici $\Rightarrow$ $p_1 \neq p_2 \neq \dots \neq p_n$

Decomposizione in fratti semplici (sviluppo di Heaviside)

$$F(s) = \frac{R_1}{s-p_1} + \frac{R_2}{s-p_2} + \dots + \frac{R_n}{s-p_n}$$

R_i : <u>residuo</u> del polo p_i

$$R_i = \lim_{s \to p_i} (s-p_i) F(s)$$

Esempio:
$$Y_\ell(s) = \frac{s+3}{(s+1)(s+2)} = \frac{R_1}{s+1} + \frac{R_2}{s+2}$$
$$p_1 = -1 \qquad p_2 = -2$$

$$R_1 = \lim_{s \to -1} (s+1)\, Y_\ell(s) = \lim_{s \to -1} \frac{s+3}{s+2} = 2$$

$$R_2 = \lim_{s \to -2} (s+2)\, Y_\ell(s) = \lim_{s \to -2} \frac{s+3}{s+1} = -1$$

$$\mathcal{L}^{-1}[F(s)] = f(t) = R_1 e^{p_1 t} + R_2 e^{p_2 t} + \dots + R_n e^{p_n t}$$
$$= \sum_{i=1}^{n} R_i e^{p_i t}$$

Esempio: mms

$$Y_\ell(s) = \frac{My(0)s + M\dot{y}(0) + \beta y(0)}{Ms^2 + \beta s + K}$$

Calcolare $y_\ell(t)$ per $M=1$, $\beta=1$, $K=1$, $y(0)=1$, $\dot{y}(0)=0$.

$$Y_\ell(s) = \frac{s+1}{s^2+s+1}$$

poli: $s = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$ $\begin{matrix} p_1 \\ p_2 \end{matrix}$

Scomposizione in fratti semplici

$$Y_\ell(s) = \frac{R_1}{s-\left(-\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)} + \frac{R_2}{s-\left(-\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)}$$

$$p_2 = \overline{p_1} \Rightarrow R_2 = \overline{R_1}$$

$$R_1 = \lim_{s \to p_1} (s-p_1) Y_\ell(s) =$$

$$= \lim_{s \to \underbrace{p_1}_{-\frac{1}{2}+j\frac{\sqrt{3}}{2}}} \left(s-\left(-\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)\right) \cdot \frac{s+1}{s^2+s+1}$$

$$= \lim_{s \to -\frac{1}{2}+j\frac{\sqrt{3}}{2}} \cancel{\left(s-\left(-\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)\right)} \cdot \frac{s+1}{\cancel{\left(s-\left(-\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)\right)}\left(s-\left(-\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)\right)}$$

$$= \frac{-\frac{1}{2}+j\frac{\sqrt{3}}{2}+1}{-\cancel{\frac{1}{2}}+j\frac{\sqrt{3}}{2}+\cancel{\frac{1}{2}}+j\frac{\sqrt{3}}{2}} = \frac{\frac{1}{2}+j\frac{\sqrt{3}}{2}}{j\sqrt{3}} =$$

$$= -\frac{1}{2\sqrt{3}}j + \frac{1}{2} = R_1 \quad \longrightarrow \quad R_2 = \frac{1}{2} + \frac{1}{2\sqrt{3}}j$$

$$y_\ell(t) = R_1 e^{p_1 t} + R_2 e^{p_2 t} =$$

$$= \left(\frac{1}{2} - \frac{1}{2\sqrt{3}}j\right) e^{\left(-\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)t} + \left(\frac{1}{2}+\frac{1}{2\sqrt{3}}j\right) e^{\left(-\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)}$$

$$R_1 = \frac{1}{2} - \frac{1}{2\sqrt{3}}j \qquad |R_1| = \sqrt{\frac{1}{4}+\frac{1}{12}} = \frac{1}{\sqrt{3}}$$

$$\angle R_1 = \arctan\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$$

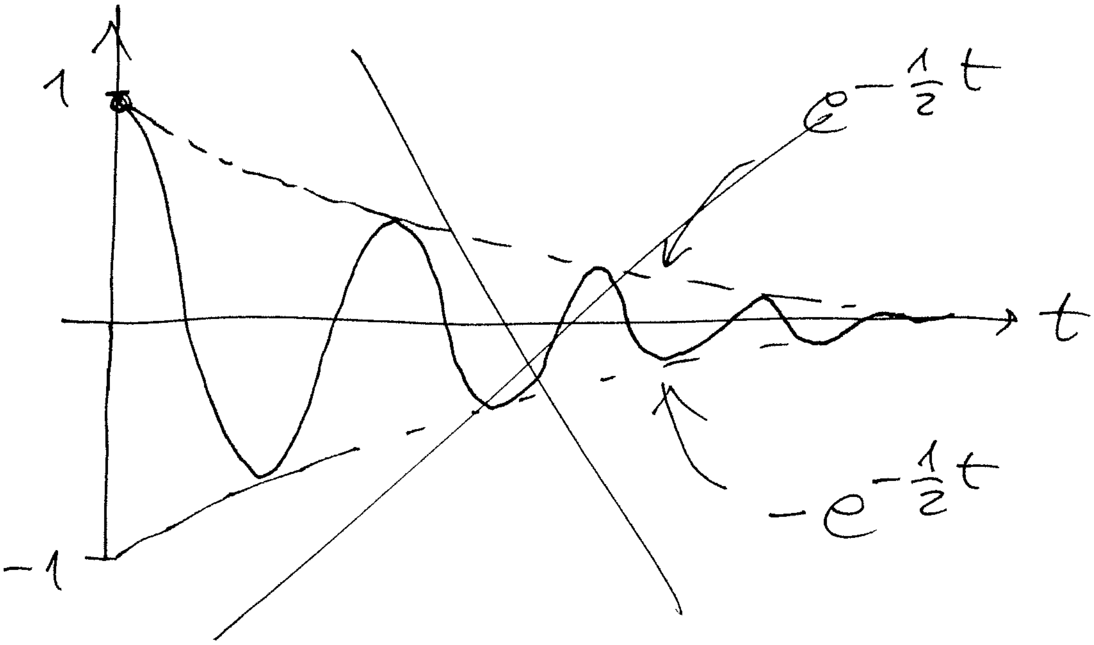
$$R_1 = \frac{1}{\sqrt{3}} e^{-j\frac{\pi}{6}} \qquad R_2 = \frac{1}{\sqrt{3}} e^{j\frac{\pi}{6}}$$

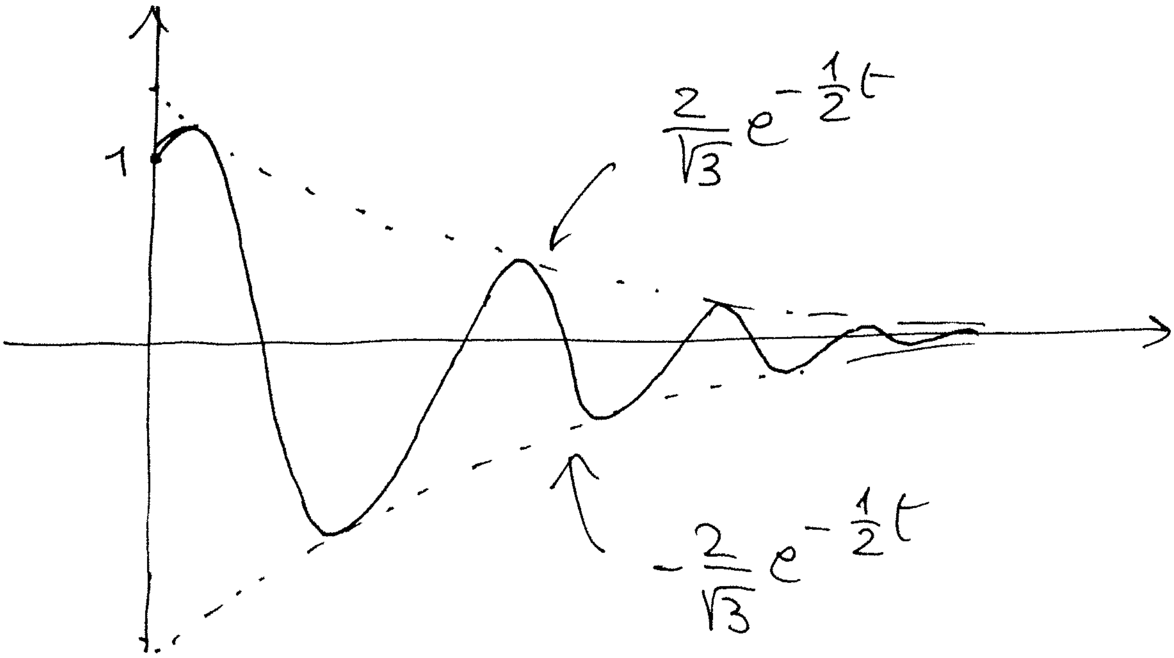
$$y_\ell(t) = \underbrace{\frac{1}{\sqrt{3}} e^{-j\frac{\pi}{6}}}_{R_1} e^{-\frac{1}{2}t} e^{j\frac{\sqrt{3}}{2}t} + \underbrace{\frac{1}{\sqrt{3}} e^{j\frac{\pi}{6}}}_{R_2} e^{-\frac{1}{2}t} e^{-j\frac{\sqrt{3}}{2}t}$$

$$= \frac{1}{\sqrt{3}} e^{-\frac{1}{2}t} \left\{ e^{j\frac{\sqrt{3}}{2}t} e^{-j\frac{\pi}{6}} + e^{-j\frac{\sqrt{3}}{2}t} e^{j\frac{\pi}{6}} \right\}$$

$$= \frac{2}{\sqrt{3}} e^{-\frac{1}{2}t} \; \frac{e^{j\left(\frac{\sqrt{3}}{2}t - \frac{\pi}{6}\right)} + e^{-j\left(\frac{\sqrt{3}}{2}t - \frac{\pi}{6}\right)}}{2}$$

$$y_\ell(t) = \frac{2}{\sqrt{3}} e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{3}}{2}t - \frac{\pi}{6}\right)$$





$\boxed{E}$. Ripetere con $\beta = 0$! (senza attrito)

In generale, una coppia di poli complessi coniugati $p_{1,2} = \sigma \pm j\omega$, di residui $R_1 = \rho e^{j\varphi}$, $R_2 = \rho e^{-j\varphi}$, dà origine ai "modi" (antitrasformata):

$$2\rho e^{\sigma t} \cos(\omega t + \varphi) = \gamma_1 \underline{e^{\sigma t} \cos \omega t} + \gamma_2 \underline{e^{\sigma t} \sin \omega t}$$

e^{pt} con $p \in \mathbb{R}$ modi "aperiodici"

$e^{\sigma t} \cos(\omega t)$
$e^{\sigma t} \sin(\omega t)$ con $p = \sigma \pm j\omega \in \mathbb{C}$
modi "pseudoperiodici"

Metodo alternativo

So che i poli sono $p_{1,2} = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$, cioè

$$Y_\ell(s) = \frac{s+1}{\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

So anche che i modi risultanti sono:

$$e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right), \quad e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right)$$

le cui trasformate sono

$$\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}, \quad \frac{\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

Devo imporre

$$\frac{s+1}{\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \gamma_1 \frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \gamma_2 \frac{\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$s+1 = \gamma_1\left(s+\frac{1}{2}\right) + \gamma_2\frac{\sqrt{3}}{2}$$

$$s+1 = \gamma_1 s + \left(\frac{1}{2}\gamma_1 + \frac{\sqrt{3}}{2}\gamma_2\right)$$

$$\gamma_1 = 1$$

$$\frac{1}{2}\gamma_1 + \frac{\sqrt{3}}{2}\gamma_2 = 1$$

$$\gamma_1 = 1$$

$$\gamma_2 = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad y_\ell(t) = \gamma_1 e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{3}}{2}t\right) + \gamma_2 e^{-\frac{1}{2}t} \sin\left(\frac{\sqrt{3}}{2}t\right)$$

$$= e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{1}{\sqrt{3}} e^{-\frac{1}{2}t} \sin\left(\frac{\sqrt{3}}{2}t\right)$$

$\boxed{E}$. Calcolare l'antitrasformata di:

$$F(s) = \frac{20}{s(s^2+2s+5)}$$

<u>Caso 2</u>. $F(s)$ strettamente propria $(n > m)$ con poli multipli:

μ_i : molteplicità algebrica del polo p_i

Forma zeri poli:

$$F(s) = \frac{N(s)}{(s-p_1)^{\mu_1} \cdot (s-p_2)^{\mu_2} \cdot \ldots \cdot (s-p_r)^{\mu_r}}$$

$\mu_i \in \mathbb{N}$, $\mu_i \geq 1$ $\qquad \mu_1 + \mu_2 + \ldots + \mu_r = n$

Decomposizione in fratti semplici

$$F(s) = F_1(s) + F_2(s) + \dots + F_r(s)$$

dove

$$F_i(s) = \frac{R_{i,1}}{s-p_i} + \frac{R_{i,2}}{(s-p_i)^2} + \dots + \frac{R_{i,\mu_i}}{(s-p_i)^{\mu_i}}$$

per $i = 1, 2, \dots, r$

Poiché sappiamo che

$$\mathcal{L}\left[\frac{t^k}{k!} e^{\alpha t} 1(t)\right] = \frac{1}{(s-\alpha)^{k+1}}$$

$$f_i(t) = \mathcal{L}^{-1}\left[F_i(s)\right] = R_{i,1} e^{p_i t} + R_{i,2}\, t e^{p_i t} + R_{i,3} \frac{t^2}{2} e^{p_i t} + \dots + R_{i,\mu_i} \frac{t^{\mu_i - 1}}{(\mu_i - 1)!} e^{p_i t}$$

I residui $R_{i,k}$ sono dati da:

$$R_{i,k} = \frac{1}{(\mu_i - k)!} \lim_{s \to p_i} \frac{d^{\mu_i - k}}{ds^{\mu_i - k}} \left[(s-p_i)^{\mu_i} F(s)\right]$$

$k = 1, 2, \dots \mu_i$

Esempio: $F(s) = \dfrac{s-6}{s^2(s+3)}$

$p_1 = \emptyset \quad \mu_1 = 2$

$p_2 = -3 \quad \mu_2 = 1$

$$F(s) = \underbrace{\frac{R_{1,1}}{s} + \frac{R_{1,2}}{s^2}}_{F_1(s)} + \underbrace{\frac{R_{2,1}}{s+3}}_{F_2(s)}$$

$$R_{1,2} = \frac{1}{(2-2)!} \lim_{s \to \emptyset} \frac{d^{2-2}}{ds^{2-2}} \left[s^2 F(s) \right] =$$

$$= \lim_{s \to \emptyset} \frac{s-6}{s+3} = -2$$

$$R_{1,1} = \frac{1}{(2-1)!} \lim_{s \to \emptyset} \frac{d^{2-1}}{ds^{2-1}} \left[s^2 F(s) \right] =$$

$$= \lim_{s \to \emptyset} \frac{d}{ds} \left(\frac{s-6}{s+3} \right) = \lim_{s \to \emptyset} \frac{s+3-(s-6)}{(s+3)^2} =$$

$$= \lim_{s \to \emptyset} \frac{9}{(s+3)^2} = 1$$

$$R_{2,1} = \lim_{s \to -3} (s+3) F(s) = \lim_{s \to -3} \frac{s-6}{s^2} = -1$$

$$F(s) = \frac{1}{s} - \frac{2}{s^2} - \frac{1}{s+3}$$

$$\mathcal{L}^{-1}[F(s)] = f(t) = 1(t) - 2t \cdot 1(t) - e^{-3t} \cdot 1(t)$$

Nel caso di coppie di poli complessi coniugati $p_{1,2} = \sigma \pm j\omega$ con molteplicità $\mu > 1$, i modi risultanti saranno

$$e^{\sigma t}\cos(\omega t),\ e^{\sigma t}\sin(\omega t),\ t e^{\sigma t}\cos(\omega t),\ t e^{\sigma t}\sin(\omega t)$$

$$t^2 e^{\sigma t}\cos(\omega t),\ t^2 e^{\sigma t}\sin(\omega t) \dots,\ t^{\mu-1} e^{\sigma t}\cos(\omega t)$$

$$t^{\mu-1} e^{\sigma t}\sin(\omega t)$$

Caso 3 : $F(s)$ propria (ma non strettamente)

$$F(s) = \frac{b_n s^n + b_{n-1} s^{n-1} + \dots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0} =$$

$$= K + \frac{\tilde{N}(s)}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0}$$

grado $\tilde{N}(s) \leq n-1$

$K = b_n$

$$\mathcal{L}^{-1}[F(s)] = K \cdot \delta(t) + \mathcal{L}^{-1}\left[\frac{\tilde{N}(s)}{s^n + a_{n-1} s^{n-1} + \dots + a_0}\right]$$

Esempio: $F(s) = \dfrac{s^2 + 5s + 3}{2s^2 + 6s + 4}$

$$F(s) = \frac{\frac{1}{2} s^2 + \frac{5}{2} s + \frac{3}{2}}{s^2 + 3s + 2} = \frac{1}{2} + \frac{\tilde{N}(s) = c_1 s + c_0}{s^2 + 3s + 2} =$$

$$= \frac{s^2 + 3s + 2 + 2c_1 s + 2c_0}{2(s^2 + 3s + 2)} = \frac{s^2 + 5s + 3}{2s^2 + 6s + 4}$$

$3 + 2c_1 = 5 \quad \rightarrow \quad c_1 = 1$

$2 + 2c_0 = 3 \quad \rightarrow \quad c_0 = \frac{1}{2}$

$$F(s) = \frac{1}{2} + \frac{s+\frac{1}{2}}{s^2+3s+2}$$

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2}\delta(t) + \mathcal{L}^{-1}\left[\frac{s+\frac{1}{2}}{s^2+3s+2}\right]$$

In alternativa

$$F(s) = \frac{\frac{1}{2}(s^2+3s+2) + s + \frac{1}{2}}{s^2+3s+2} =$$

$$= \frac{1}{2} + \frac{s+\frac{1}{2}}{s^2+3s+2}$$