Problema 3, Elaborato 2.

1) $x_1(k) = y(k)$
$x_2(k) = y(k+1)$

$$x_1(k+1) = y(k+1) = x_2(k)$$

$$x_2(k+1) = y(k+2) = -\frac{1}{2}\underbrace{y(k+1)}_{x_2(k)} + \frac{1}{2}\underbrace{y(k)}_{x_1(k)} + u(k)$$

$$y(k) = x_1(k)$$

$$A = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad D = [0]$$

2) $y(0) = -\frac{1}{2}\underbrace{y(-1)}_{0} + \frac{1}{2}\underbrace{y(-2)}_{1} + \cancel{u(-2)} = \frac{1}{2}$

$$y(1) = -\frac{1}{2}\underbrace{y(0)}_{\frac{1}{2}} + \frac{1}{2}\underbrace{y(-1)}_{0} + \cancel{u(-1)} = -\frac{1}{4}$$

$$x(0) = \begin{bmatrix} y(0) \\ y(1) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{4} \end{bmatrix}$$

$$Y_\ell(z) = C(zI-A)^{-1} \cdot z \cdot x(0) =$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} z & -1 \\ -\frac{1}{2} & z+\frac{1}{2} \end{bmatrix}^{-1} \cdot z \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{4} \end{bmatrix} =$$

$$= \frac{\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} z+\frac{1}{2} & 1 \\ \frac{1}{2} & z \end{bmatrix} z \cdot \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{4} \end{bmatrix}}{z^2 + \frac{1}{2}z - \frac{1}{2}} =$$

$$= \frac{\begin{bmatrix} z+\frac{1}{2} & 1 \end{bmatrix} \cdot z \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{4} \end{bmatrix}}{z^2+\frac{1}{2}z-\frac{1}{2}} = \frac{z\left(\frac{1}{2}z+\cancel{\frac{1}{4}}-\cancel{\frac{1}{4}}\right)}{z^2+\frac{1}{2}z-\frac{1}{2}} =$$

$$= \frac{\frac{1}{2}z^2}{z^2+\frac{1}{2}z-\frac{1}{2}}$$

$$\frac{Y_\ell(z)}{z} = \frac{\frac{1}{2}z}{z^2+\frac{1}{2}z-\frac{1}{2}} = \frac{R_1}{z-\frac{1}{2}} + \frac{R_2}{z+1}$$

$$z^2+\frac{1}{2}z-\frac{1}{2}=0 \qquad z_{1,2} = \frac{-\frac{1}{2}\pm\sqrt{\frac{1}{4}+2}}{2} = \left\langle \begin{array}{l} \frac{1}{2} \\ -1 \end{array} \right.$$

$$\frac{R_1(z+1)+R_2\left(z-\frac{1}{2}\right)}{\left(z-\frac{1}{2}\right)(z+1)} = \frac{\frac{1}{2}z}{z^2+\frac{1}{2}z-\frac{1}{2}}$$

$$\underline{R_1 z} + \underline{\underline{R_1}} + \underline{R_2 z} - \underline{\underline{\tfrac{1}{2}R_2}} = \underline{\tfrac{1}{2}z} \Rightarrow \begin{cases} R_1+R_2 = \frac{1}{2} \\ R_1 - \frac{1}{2}R_2 = 0 \end{cases}$$

$$R_2 = \frac{1}{3} \quad R_1 = \frac{1}{6}$$

$$Y_\ell(z) = \frac{1}{6}\cdot\frac{z}{z-\frac{1}{2}} + \frac{1}{3}\,\frac{z}{z+1}$$

$$y_\ell(k) = \left\{\frac{1}{6}\cdot\left(\frac{1}{2}\right)^k + \frac{1}{3}\cdot(-1)^k\right\}\mathbb{1}(k)$$

Approccio alternativo: fare direttamente la trasformata Zeta dell'equazione i/o, tenendo conto delle condizioni iniziali (fare per esercizio!)

3) $$G(z) = \frac{1}{z^2+\frac{1}{2}z-\frac{1}{2}}$$

$$Y_f(z) = G(z)\cdot\frac{z}{z-1} = \frac{1}{z^2+\frac{1}{2}z-\frac{1}{2}}\cdot\frac{z}{z-1}$$

...

4) $$u(k) = (-1)^k \cdot \mathbb{1}(k) \longrightarrow U(z) = \frac{z}{z+1}$$

$$Y_f(z) = G(z)\cdot U(z) = \frac{1}{z^2+\frac{1}{2}z-\frac{1}{2}}\cdot\frac{z}{z+1}$$

$$(z-\tfrac{1}{2})(z+1)$$

$\frac{1}{2}$, $\mu = 1$ $\rightarrow$ modo $\left(\frac{1}{2}\right)^k$

-1, $\mu = 2$ $\rightarrow$ modi $(-1)^k$, $\underline{k\cdot(-1)^k}$

$\longrightarrow$ $y_f(k)$ diverge

Problema 4. Elaborato 2

1) $W(z) = \mathcal{Z}\{w(k)\} = \sum_{k=0}^{+\infty} w(k)\cdot z^{-k} =$

$= 1\cdot z^{-0} \underline{- 1\cdot z^{-1}} + 3\sigma^2 z^{-2} \underline{- 3\sigma^2 z^{-3}} + 9\sigma^4 z^{-4}$
$\underline{- 9\sigma^4 z^{-5}} + 27\sigma^6 z^{-6} \underline{- 27\sigma^6 z^{-7}} + \cdots + \cdots$

$= \{1 + 3\sigma^2 z^{-2} + 9\sigma^4 z^{-4} + 27\sigma^6 z^{-6} + \cdots\} +$
$- z^{-1}\{\underbrace{1 + 3\sigma^2 z^{-2} + 9\sigma^4 z^{-4} + 27\sigma^6 z^{-6} + \cdots}\} =$

$= (1 - z^{-1})\{\;\cdots\;\}$

$= (1 - z^{-1}) \cdot \sum_{k=0}^{+\infty} (\sqrt{3}\sigma z^{-1})^{2k} =$

$= (1 - z^{-1}) \cdot \sum_{k=0}^{+\infty} (3\sigma^2 z^{-2})^{k} =$

$= (1 - z^{-1}) \cdot \frac{1}{1 - 3\sigma^2 z^{-2}} = \frac{z^2 - z}{z^2 - 3\sigma^2}$

2)
poli: $z^2 - 3\sigma^2 = 0 \qquad z = \pm\sqrt{3}\sigma \qquad (\sigma > 0)$

modi: $(\sqrt{3}\sigma)^k \quad (-\sqrt{3}\sigma)^k$

$|\sqrt{3}\sigma| < 1 \quad \Rightarrow \quad |\sigma| < \frac{1}{\sqrt{3}}$