Elaborato 7. Problema 4.

1. $$A = \begin{bmatrix} \frac{1}{2} & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 0 \end{bmatrix}$$

$$G(z) = C(zI - A)^{-1}B + D =$$

$$= \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} z-\frac{1}{2} & -1 & 0 \\ 0 & z-1 & 0 \\ 0 & 0 & z-1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} =$$

$$= \frac{\begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} (z-1)^2 & z-1 & 0 \\ 0 & (z-\frac{1}{2})(z-1) & 0 \\ 0 & 0 & (z-\frac{1}{2})(z-1) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}{(z-\frac{1}{2})(z-1)^2} =$$

$$= \frac{\begin{bmatrix} (z-1)^2 & (z-1) & (z-\frac{1}{2})(z-1) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}{(z-\frac{1}{2})(z-1)^2} =$$

$$= \frac{(z-1) + (z-\frac{1}{2})(z-1)}{(z-\frac{1}{2})(z-1)^2} = \frac{(z-1)(z+\frac{1}{2})}{(z-\frac{1}{2})(z-1)^2} = \frac{z+\frac{1}{2}}{(z-\frac{1}{2})(z-1)}$$

2. Risposta impulsiva

$$y_{imp}(k) = \mathcal{Z}^{-1}\left[\frac{z+\frac{1}{2}}{(z-\frac{1}{2})(z-1)}\right] = \mathcal{Z}^{-1}\left[\frac{R_1}{z-\frac{1}{2}} + \frac{R_2}{z-1}\right]$$

$$R_1 = \lim_{z \to \frac{1}{2}} (z-\tfrac{1}{2}) \cdot G(z) = \lim_{z \to \frac{1}{2}} \frac{z+\frac{1}{2}}{z-1} = \frac{1}{-\frac{1}{2}} = -2$$

$$R_2 = \lim_{z \to 1} (z-1) G(z) = \lim_{z \to 1} \frac{z+\frac{1}{2}}{z-\frac{1}{2}} = \frac{\frac{3}{2}}{\frac{1}{2}} = 3$$

$$y_{imp}(k) = -2 \cdot \left(\frac{1}{2}\right)^{k-1} \cdot 1\!\!1(k-1) + 3 \cdot 1\!\!1(k-1) = \left(3 - 2\left(\frac{1}{2}\right)^{k-1}\right) \cdot 1\!\!1(k-$$

3. Scegliendo $u(k) = 1\!\!1(k)$, $y_f(k)$ conterrà il modo divergente $k \cdot 1\!\!1(k)$ $\Rightarrow$ Il sistema <u>non</u> è stabile in senso ILUL (c'è un polo in $G(z)$ con modulo 1)

Stabilità interna: autovalori di $A \Rightarrow \lambda_1 = \frac{1}{2}$ $\lambda_2 = 1$ $\lambda_3 = 1$

$\lambda_{2,3} = 1$ ha molteplicità algebrica 2

molt. geometrica $= \dim \ker(\lambda I - A) = \dim \ker(I - A)$

$$= \dim \ker \begin{bmatrix} \frac{1}{2} & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 2$$

$\Rightarrow$ Il sistema è marginalmente stabile.

4. Raggiungibilità

$$Q = [B \;\; AB \;\; A^2B] = \begin{bmatrix} 0 & 1 & \frac{3}{2} \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

rank $Q = 2$ $\Rightarrow$ il sistema non è completamente raggiungibile

5. $$X_1^R = \operatorname{Im} B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \qquad \bar{x} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \notin X_1^R$$

$$X_2^R = \operatorname{Im}[B \;\; AB] = \operatorname{Im} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad \bar{x} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \in X_2^R$$

$$x_f(T) = [B \;\; AB \; \ldots \; A^{T-1}B] \begin{bmatrix} u(T-1) \\ u(T-2) \\ \vdots \\ u(0) \end{bmatrix} \Rightarrow \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix}$$

$\Rightarrow u(0) = 3$, $u(1) = -2$

6. $$x_f(T) - A^T x(\emptyset) = \begin{bmatrix} B & AB & \cdots & A^{T-1}B \end{bmatrix} \begin{bmatrix} u(T-1) \\ u(T-2) \\ \vdots \\ u(0) \end{bmatrix}$$

$$x_f(T) = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \qquad x(\emptyset) = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$$

$T=1$ $$\underbrace{\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} - A \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}} = B \cdot u(\emptyset) \quad \left(\in \operatorname{Im} B \quad ? \right)$$

$$\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \notin \operatorname{Im} B = \mathcal{L}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$T=2$ $$\underbrace{\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} - A^2 \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}} \in \operatorname{Im} \begin{bmatrix} B & AB \end{bmatrix}$$

$$\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix}$$

$$u(\emptyset) = \emptyset, \quad u(1) = -1.$$

7. $$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

base di $\mathcal{X}^R$ — completa la base di $\mathbb{R}^3$

$$\bar{A} = T^{-1} A T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{1}{2} & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \left[\begin{array}{cc|c} \frac{1}{2} & 1 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] = \begin{pmatrix} A_r & A_{r\bar{r}} \\ 0 & A_{\bar{r}} \end{pmatrix}$$

autovalore non raggiungibile

$$\bar{B} = T^{-1}B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \hline 0 \end{bmatrix} = \begin{bmatrix} B_r \\ \hline \emptyset \end{bmatrix}$$

$$\bar{C} = CT = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

Il sistema non è stabilizzabile (autovalore non raggiungibile con modulo 1).

8. Osservabilità

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ \frac{1}{2} & 1 & 1 \\ \frac{1}{4} & \frac{3}{2} & 1 \end{bmatrix}$$

$$\det \mathcal{O} = 1 - \frac{3}{2} + \frac{3}{4} - \frac{1}{4} = 0$$

Il sistema non è completamente osservabile!

$$X^{no} = \ker \mathcal{O}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ \frac{1}{2} & 1 & 1 \\ \frac{1}{4} & \frac{3}{2} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \qquad \left.\begin{array}{l} x + z = 0 \\ \frac{1}{2}x + y + z = 0 \\ \frac{1}{4}x + \frac{3}{2}y + z = 0 \end{array}\right\}$$

$$\left.\begin{array}{l} \frac{1}{2}x + y - x = 0 \\ z = -x \end{array}\right\} \qquad \left.\begin{array}{l} y = \frac{1}{2}x \\ z = -x \end{array}\right\}$$

$$X^{no} = L \begin{pmatrix} 1 \\ \frac{1}{2} \\ -1 \end{pmatrix}$$

9.

$$T = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & +1 \end{bmatrix}$$

Completiamo la base di $\mathbb{R}^3$

base di $\mathcal{X}^{no}$

$$\bar{A} = T^{-1} A T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & 1 & 1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} = \left[\begin{array}{cc|c} \frac{1}{2} & 1 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} A_o & \emptyset \\ \hline A_{o\bar{o}} & A_{\bar{o}} \end{array}\right]$$

autovalore non osservabile

$\Rightarrow$ Il sistema <u>non</u> è rivelabile.

Problema 6.

1. $$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -12 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 10 & 0 \end{bmatrix}$$

$$G(s) = C(sI-A)^{-1}B = \begin{bmatrix} 2 & 10 & 0 \end{bmatrix} \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 10 & 12 & s+3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{\begin{bmatrix} 2 & 10 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ s \\ s^2 \end{bmatrix}}{s(s^2+3s+12)+10} = \frac{2+10s}{\underset{\overset{\shortparallel}{(s+1)(s^2+2s+10)}}{s^3+3s^2+12s+10}}$$

zero : $-\frac{1}{5}$ poli : -1, $-1 \pm j3$

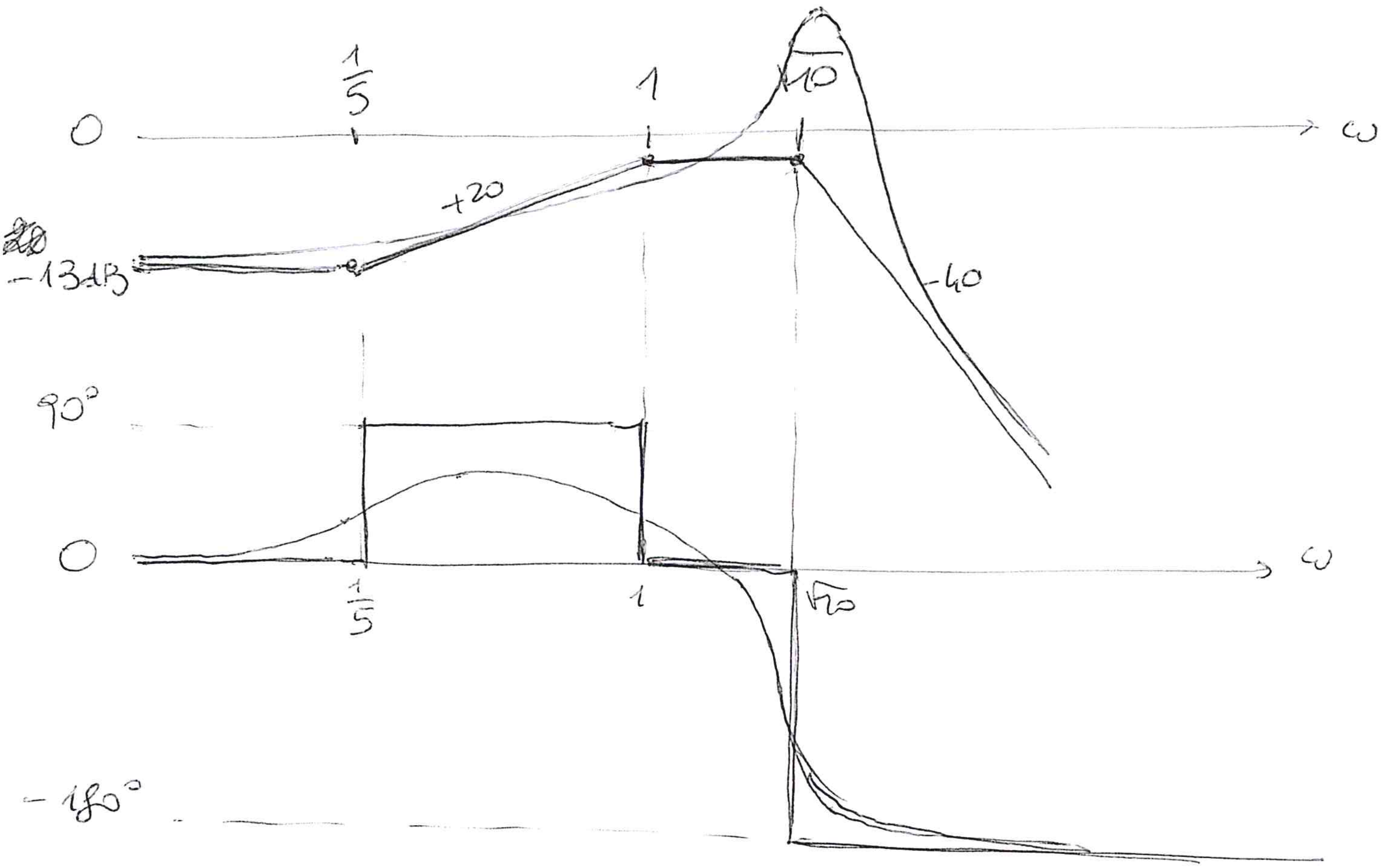
$$\begin{array}{r|rrr|r} & 1 & 3 & 12 & 10 \\ -1 & & -1 & -2 & -10 \\ \hline & 1 & 2 & 10 & 0 \end{array}$$

$$G(s) = \frac{2(1+5s)}{(1+s)\cdot 10\left(1+\frac{2}{10}s+\frac{1}{10}s^2\right)}$$

zero reale : $\hat{\tau} = \frac{1}{5}$ polo reale : $\tau = 1$ poli complessi $\omega_n = \sqrt{10}$, $\zeta = \frac{1}{\sqrt{10}}$

$$\frac{2\cdot\zeta}{\omega_n} = \frac{2}{10} \qquad k_B = \frac{1}{5} \qquad 20\log_{10}\frac{1}{5} \simeq -13\,dB$$

Diagrammi di Bode:



2. $Q = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & -3 & -3 \end{bmatrix}$ rank $Q = 3$

$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 2 & 10 & 0 \\ 0 & 2 & 10 \\ -100 & -120 & -28 \end{bmatrix}$

$\det \mathcal{O} = 2(-56 - 1200) - 100 \cdot 100 \neq 0$ rank $\mathcal{O} = 3$

Sistema completamente raggiungibile e osservabile

3. $\det(\lambda I - A - BF) = (\lambda+1)^3$

$F = [f_1 \quad f_2 \quad f_3]$

$$A + BF = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -12 & -3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot (f_1 \ f_2 \ f_3) =$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10+f_1 & -12+f_2 & -3+f_3 \end{pmatrix}$$

$$\det(\lambda I - A - BF) = \det \begin{pmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 10-f_1 & 12-f_2 & \lambda+3-f_3 \end{pmatrix} =$$

$$= \lambda\left[\lambda^2 + (3-f_3)\lambda + 12 - f_2\right] + 10 - f_1 =$$

$$= \lambda^3 + (3-f_3)\lambda^2 + (12-f_2)\lambda + (10-f_1)$$

$$\lambda^3 + (3-f_3)\lambda^2 + (12-f_2)\lambda + (10-f_1) = \lambda^3 + 3\lambda^2 + 3\lambda + 1$$

$$\begin{matrix} 3 - f_3 = 3 \\ 12 - f_2 = 3 \\ 10 - f_1 = 1 \end{matrix} \implies \begin{matrix} f_3 = 0 \\ f_2 = 9 \\ f_1 = 9 \end{matrix} \implies F = (9 \ 9 \ 0)$$

4. $\tilde{x}(t) = x(t) - \hat{x}(t) \longrightarrow \dot{\tilde{x}}(t) = (A - LC)\tilde{x}(t)$

$\det(\lambda I - A + LC) = (\lambda+4)(\lambda+5)(\lambda+6)$

$$A - LC = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -12 & -3 \end{pmatrix} - \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix} (2 \quad 10 \quad 0) =$$

$$= \begin{pmatrix} -2l_1 & 1 - 10l_1 & 0 \\ -2l_2 & -10l_2 & 1 \\ -10-2l_3 & -12-10l_3 & -3 \end{pmatrix}$$

$$\det\begin{pmatrix} \lambda+2l_1 & 10l_1-1 & 0 \\ 2l_2 & \lambda+10l_2 & -1 \\ 10+2l_3 & 12+10l_3 & \lambda+3 \end{pmatrix} = \underbrace{(\lambda+4)(\lambda+5)(\lambda+6)}$$

$$(\lambda+2l_1)\left[\lambda^2+3\lambda+10l_2\lambda+30l_2+12+10l_3\right]+$$
$$+(1-10l_1)\left[2l_2\lambda+6l_2+10+2l_3\right]=$$

$$=\lambda^3+\lambda^2\left[2l_1+3+10l_2\right]+\lambda\left[30l_2+12+10l_3+6l_1+\cancel{20l_1}l_2\right.$$
$$\left.+2l_2-\cancel{20l_1}l_2\right]+\left[\cancel{60l_1}l_2+24l_1+\cancel{20l_1}l_3+6l_2+10+2l_3\right.$$
$$\left.-\cancel{60l_1}l_2-100l_1-\cancel{20l_1}l_3\right]=$$

$$=(\lambda^2+9\lambda+20)(\lambda+6)=\lambda^3+15\lambda^2+74\lambda+120$$

$$\left.\begin{aligned} 2l_1+10l_2+3&=15 \\ 6l_1+32l_2+10l_3+12&=74 \\ -76l_1+6l_2+2l_3+10&=120 \end{aligned}\right\} \ldots$$

Matlab:

place $(A', C', [-4\ \ -5\ \ -6]) \rightarrow L'$