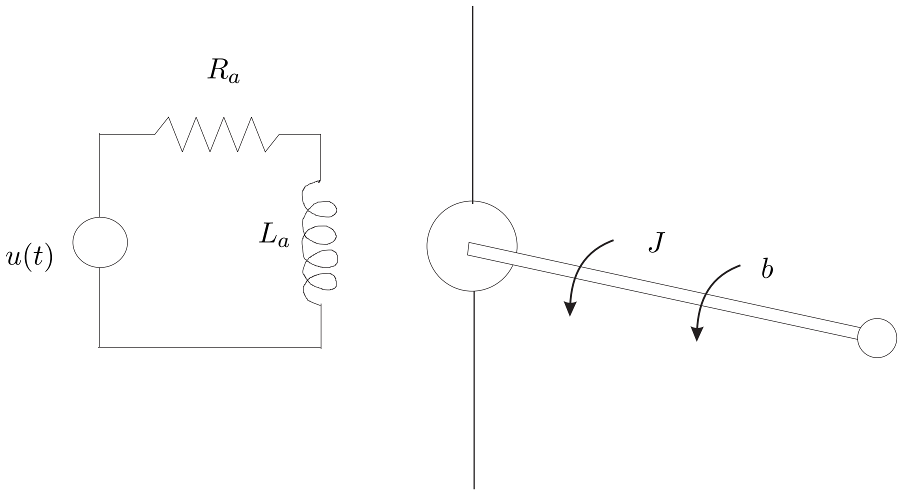
**State Estimation and Filtering**
**Labsession on Kalman Filter**

The behavior of the DC motor with field control represented in the figure is described by the following discrete-time dynamic model:




$$x_1(t+1) = x_1(t) - \frac{T_c}{J}(bx_1(t) - K_m x_2(t))$$
$$x_2(t+1) = x_2(t) - \frac{T_c}{L_a}(K_m x_1(t) + R_a x_2(t)) + \frac{T_c}{L_a}(u(t) + w(t))$$

where

- $x_1(t)$ is the angular velocity of the rotating shaft;
- $x_2(t)$ is the current flowing in the field circuit;
- $u(t)$ is the voltage applied to the field circuit;
- $w(t)$ is a disturbance on the voltage applied to the field circuit, modeled as a white stochastic process with zero mean and variance $Q$;
- $R_a, L_a$ are the resistance and inductance of the circuit;
- $J, b$ are respectively the moment of inertia and the friction coefficient of the shaft;
- $K_m$ is the field gain of the motor;
- $T_c$ is the sampling time.

A measurement of the angular velocity of the shaft is available, i.e.

$$y(t) = x_1(t) + v(t)$$

where $v(t)$ is the measurement noise, modeled as a white stochastic process with zero mean and variance $R$.
In the file `data_kf_dcmotor.npz` (`.mat` for Matlab), an input-output data set $\{u(t), y(t)\}$, for $t = 0, 1 \ldots, N$, is provided. Such data have been generated from an experiment on the above system with

the following parameters: $J = 0.01$ $(kgm^2/s^2)$, $b = 0.1$ $(Nms)$, $K_m = 0.01$ $(Nm/A)$, $R_a = 1$ $(\Omega)$, $L_a = 1$ $(H)$, $T_c = 0.01$ $(s)$. The variances of the process disturbance and measurement noise are $Q = 1$ $(V^2)$ and $R = 0.01$ $(rad^2/s^2)$, respectively.

a) Design and implement a linear Kalman filter which, by using the data $y(t), u(t)$, provides an estimate of the time evolution of the state variables $x_i(t)$, $i = 1, 2$, for all time instants $t = 0, 1, \ldots, N$. Compare the estimated states with the corresponding true values. Evaluate the consistency of the filter by comparing the estimation error of each state variable $x_i(t) - \hat{x}_i(t|t)$ with the corresponding confidence intervals $\pm 3\sqrt{P_{ii}(t|t)}$. Compute the root-mean-square error $\sqrt{\frac{1}{N+1}\sum_{t=0}^{N}\{x_i(t) - \hat{x}_i(t|t)\}^2}$, for $i = 1, 2$. Analyze the behavior of the filter for different initial conditions $\hat{x}(0|-1)$ and $P(0|-1)$. Compare the obtained state estimates with those provided by the asymptotic Kalman filter.

b) Now assume that the measurement of the angular velocity is affected by a constant bias $b$, i.e.,

$$y_b(t) = x_1(t) + b + v(t)$$

Design a Kalman Filter which is able to simultaneously estimate the state variables $x_1(t)$, $x_2(t)$ and the bias $b$, by using the data $y_b(t), u(t)$. Evaluate the consistency of the filter by comparing the estimation error of each estimated variable with the corresponding $3\sigma$ confidence intervals. Compare the results with those obtained by a standard Kalman Filter which does not estimate the bias term. Compare the RMSE obtained by both filters. Analyze the behavior of the filter for different initial conditions $\hat{x}(0|-1)$ and $P(0|-1)$. Discuss the role of the variance $\sigma_b^2$ of the process disturbance affecting the dynamic equation of the variable corresponding to the bias term.

Data available in the file `data_kf_dcmotor.npz`:
`Y`: vector of measurements $y(t)$, $t = 0, \ldots, N$
`U`: vector of inputs $u(t)$, $t = 0, \ldots, N$
`Ybias`: vector of measurements $y_b(t)$, $t = 0, \ldots, N$, affected by bias error
`X`: matrix of true states $x_i(t)$, $i = 1, 2$, $t = 0, \ldots, N$, to be used only for the comparisons with the estimated states (the entry $(t, i)$ of the matrix corresponds to $x_i(t-1)$)
`bias`: the true bias error $b$, to be used only for the comparisons with the estimated bias.