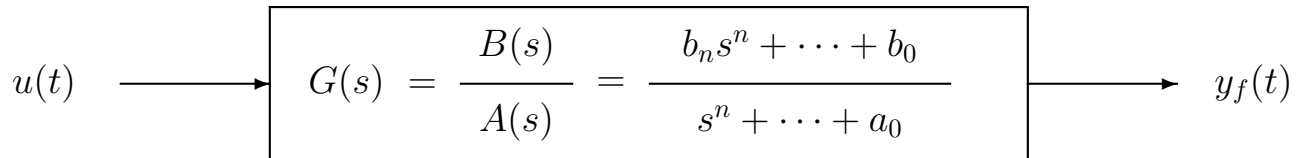


RISPOSTA FORZATA SISTEMI LINEARI STAZIONARI



- Classe di funzioni di ingresso.

$$\mathcal{U} := \left\{ u(\cdot) : U(s) = \frac{Q(s)}{P(s)} = \frac{\prod_{i=1}^l (s - z_i)}{\prod_{i=1}^r (s - p_i)}, \quad l \leq r, \quad A(p_i) \neq 0, \quad i = 1, \dots, r \right\}$$

- Forma di $Y_f(s)$ (caso p_i distinti)

$$Y_f(s) = G(s)U(s) = H(s) + \sum_{i=1}^n \frac{k_i}{s - p_i}$$

- Scomposizione risposta forzata: $y_f(t) = y_f^G(t) + y_f^U(t)$.

– Parte dipendente dai poli di $G(s)$ (“transitorio”).

$$y_f^G(t) = \mathcal{L}^{-1}\{H(s)\}$$

– Parte dipendente dai poli di $U(s)$ (“regime permanente”).

$$y_f^U(t) = \mathcal{L}^{-1}\left\{ \sum_{i=1}^n \frac{k_i}{s - p_i} \right\}$$

RAPPRESENTAZIONI FUNZIONE DI TRASFERIMENTO

- Funzione di trasferimento razionale fratta

$$G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots a_1 s + a_0}$$

- Forma poli-zero

$$G(s) = K' \frac{(s - z_1)(s - z_2) \dots (s - z_m)}{(s - p_1)(s - p_2) \dots (s - p_n)} \quad K' = \frac{b_m}{a_n}$$

- Forma costanti di tempo o di Bode

$$G(s) = \frac{K_B}{s^h} \frac{(1 + \tau'_1 s) \dots (1 + \tau'_{zr} s) (1 + 2\zeta'_1 \frac{s}{\omega'_{n_1}} + \frac{s^2}{\omega'^2_{n_1}}) \dots (1 + 2\zeta'_{zc} \frac{s}{\omega'_{n_{zc}}} + \frac{s^2}{\omega'^2_{n_{zc}}})}{(1 + \tau_1 s) \dots (1 + \tau_{pr} s) (1 + 2\zeta_1 \frac{s}{\omega_{n_1}} + \frac{s^2}{\omega^2_{n_1}}) \dots (1 + 2\zeta_{pc} \frac{s}{\omega_{n_{pc}}} + \frac{s^2}{\omega^2_{n_{pc}}})}$$

$$K_B = K' \tau_1 \dots \tau_{pr} \omega'^2_{n_1} \dots \omega'^2_{n_{zc}} \tau'^{-1}_1 \dots \tau'^{-1}_{zr} \omega^{-2}_{n_1} \dots \omega^{-2}_{n_{pc}} \quad \text{guadagno di Bode}$$

$$\tau'_i = -\sigma'^{-1}_i \quad \text{zeri reali } z_i = \sigma'_i$$

$$\zeta'_i = -\sigma'_i [\sigma'^2_i + \omega'^2_i]^{-1/2}; \quad \omega'_{n_i} = \sqrt{\sigma'^2_i + \omega'^2_i} \quad \text{zeri complessi } z_i = \sigma'_i + j\omega'_i$$

$$\tau_i = -\sigma_i^{-1} \quad \text{poli reali } p_i = \sigma_i$$

$$\zeta_i = -\sigma_i [\sigma_i^2 + \omega_i^2]^{-1/2}; \quad \omega_{n_i} = \sqrt{\sigma_i^2 + \omega_i^2} \quad \text{poli complessi } p_i = \sigma_i + j\omega_i$$

RISPOSTA AL GRADINO DEI SISTEMI DEL PRIMO ORDINE

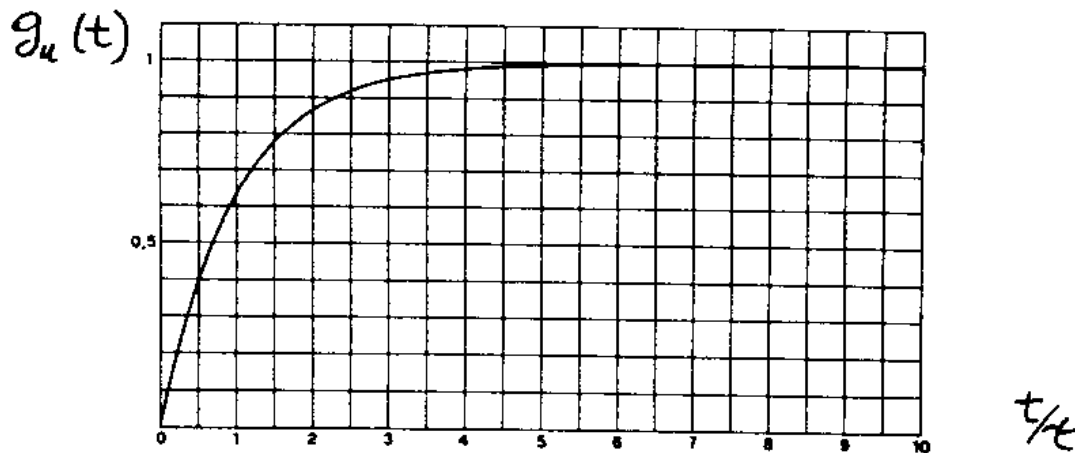
- Sistema del primo ordine.

$$G(s) = \frac{K}{1 + \tau s}; \quad g(t) = \frac{K}{\tau} e^{-t/\tau}$$

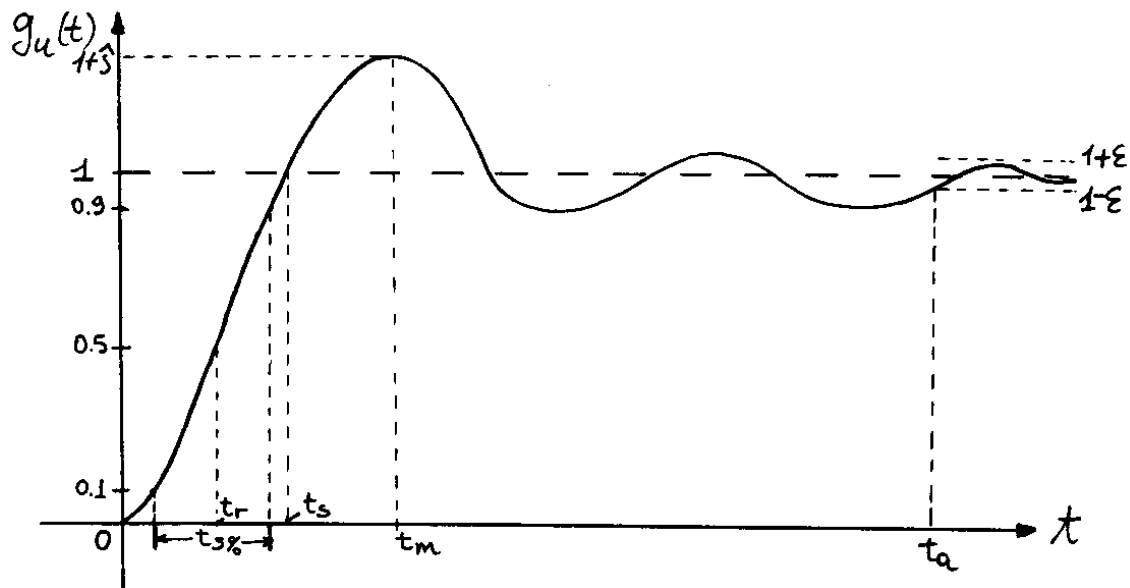
- Risposta al gradino ($K = 1$)

$$g_u(t) = \mathcal{L}^{-1} \left\{ \frac{1}{s(1 + \tau s)} \right\} = \mathcal{L}^{-1} \left\{ \frac{-\tau}{(1 + \tau s)} \right\} + \mathcal{L}^{-1} \left\{ \frac{1}{s} \right\} = -e^{-t/\tau} + 1$$

- Andamento nel tempo ($\tau > 0$).



PARAMETRI CARATTERISTICI RISPOSTA AL GRADINO



- Massima sovraelongazione: \hat{s}
- Tempo di ritardo: t_r
- Tempo di salita: $t_{s\%}$ e t_s
- Tempo di assestamento: t_a
- Istante di massima sovraelongazione: t_m

RISPOSTA AL GRADINO: SISTEMI SECONDO ORDINE

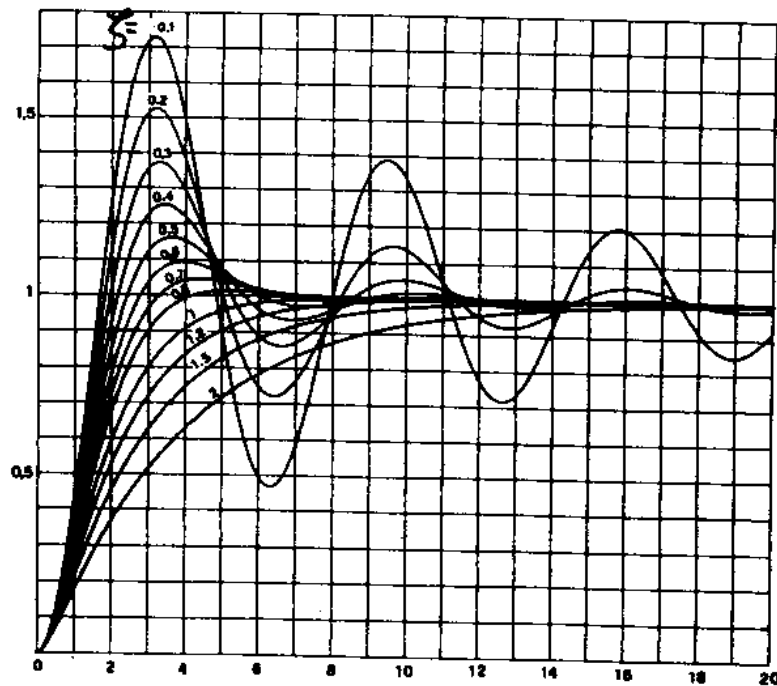
- Sistema del secondo ordine

$$G(s) = \frac{1}{1 + 2\zeta \frac{s}{\omega_n} + \frac{s^2}{\omega_n^2}}$$

- Risposta al gradino

$$\begin{aligned} g_u(t) &= \mathcal{L}^{-1} \left\{ \frac{1}{1 + 2\zeta \frac{s}{\omega_n} + \frac{s^2}{\omega_n^2}} \frac{1}{s} \right\} = \mathcal{L}^{-1} \left\{ -\frac{2\zeta}{\omega_n} \frac{1 + \frac{s}{2\zeta\omega_n}}{1 + 2\zeta \frac{s}{\omega_n} + \frac{s^2}{\omega_n^2}} \right\} + \mathcal{L}^{-1} \left\{ \frac{1}{s} \right\} = \\ &= -\frac{1}{\sqrt{1 - \zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1 - \zeta^2} t + \arctan \frac{\sqrt{1 - \zeta^2}}{\zeta}) + 1 \end{aligned}$$

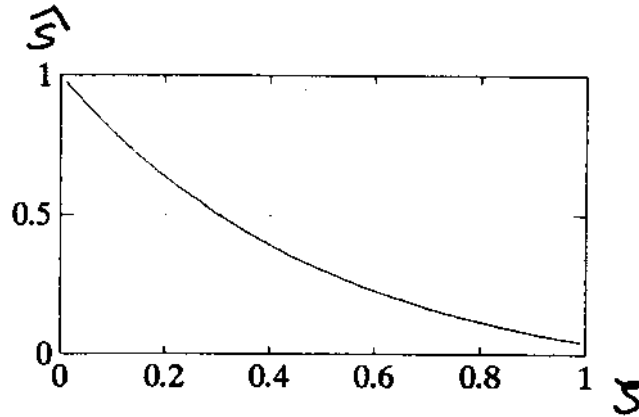
$g_u(t)$



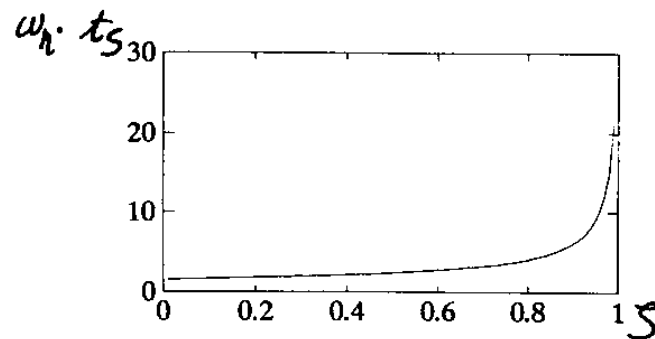
$\omega_n t$

RISPOSTA AL GRADINO: SISTEMI SECONDO ORDINE

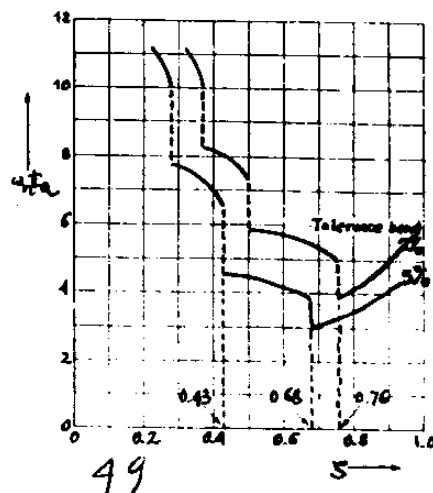
- Massima sovraelongazione: $\hat{s} = \exp(-\pi\zeta/\sqrt{1-\zeta^2})$



- Tempo di salita: $t_s = \omega_n^{-1}[1 - \zeta^2]^{-1/2}[\pi - \arctan \zeta^{-1}\sqrt{1 - \zeta^2}]$



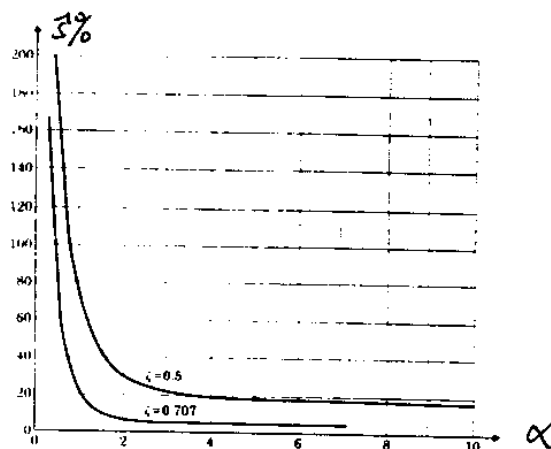
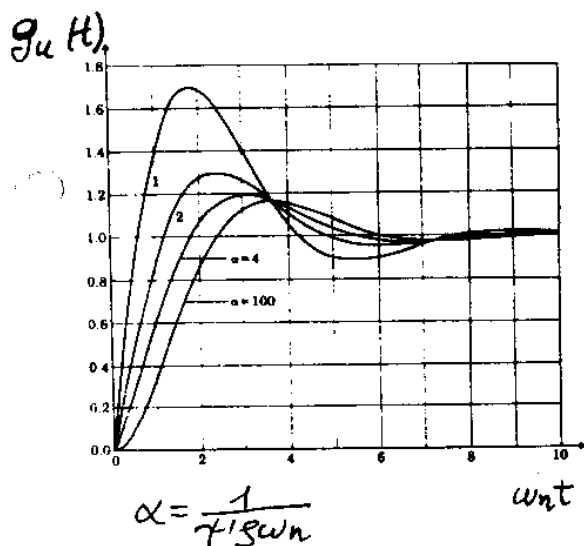
- Tempo di assestamento



RISPOSTA AL GRADINO: SISTEMI SECONDO ORDINE

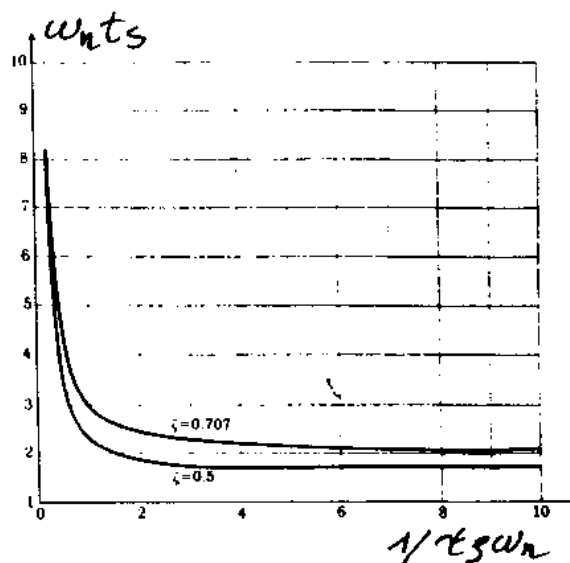
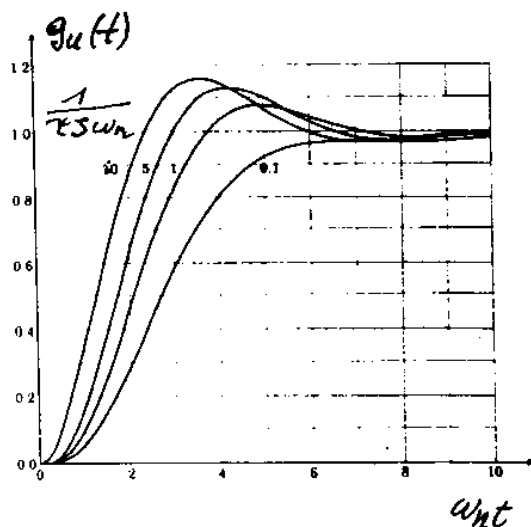
- Aggiunta di uno zero

$$G(s) = \frac{1 + \tau' s}{1 + 2\zeta \frac{s}{\omega_n} + \frac{s^2}{\omega_n^2}}$$

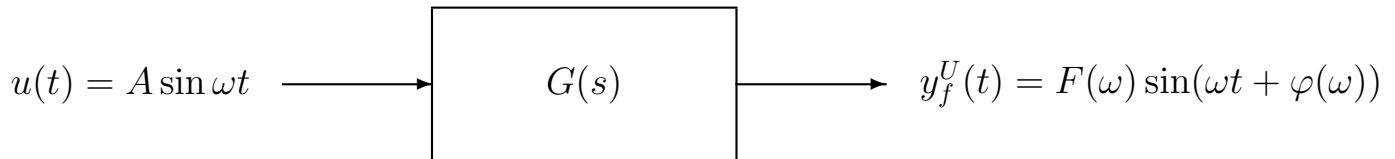


- Aggiunta di un polo

$$G(s) = \frac{1}{(1 + \tau s)(1 + 2\zeta \frac{s}{\omega_n} + \frac{s^2}{\omega_n^2})}$$



RISPOSTA AD UNA SINUSOIDE (RISPOSTA IN FREQUENZA)



- Valutazione risposta forzata

$$y(t) = \mathcal{L}^{-1} \left\{ G(s) \frac{A\omega}{s^2 + \omega^2} \right\} = \underbrace{\mathcal{L}^{-1} \{ H(s) \}}_{y_f^G(t)} + \underbrace{\mathcal{L}^{-1} \left\{ \frac{k_-}{s - j\omega} + \frac{k_+}{s + j\omega} \right\}}_{y_f^U(t)}$$

- Teorema della risposta in frequenza ($G(s)$ con poli a parte reale < 0)

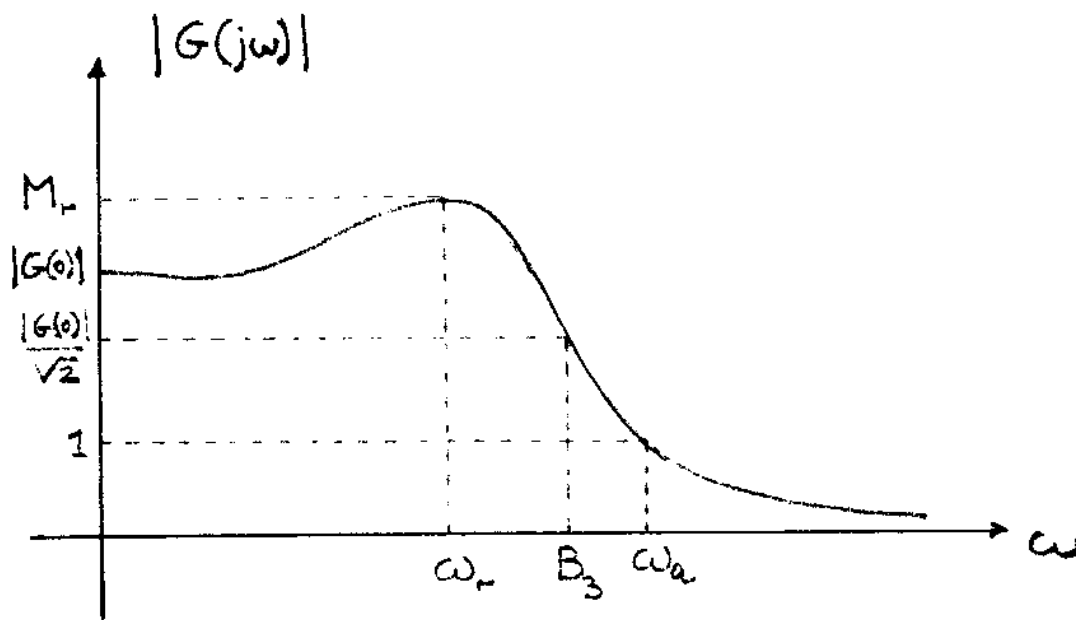
$$F(\omega) = A|G(j\omega)|; \quad \varphi(\omega) = \arg G(j\omega)$$

- Risposta in frequenza \leftrightarrow risposta impulsiva

$$G(j\omega) = \int_0^\infty g(t) e^{-j\omega t} dt$$

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^\infty G(j\omega) e^{j\omega t} d\omega$$

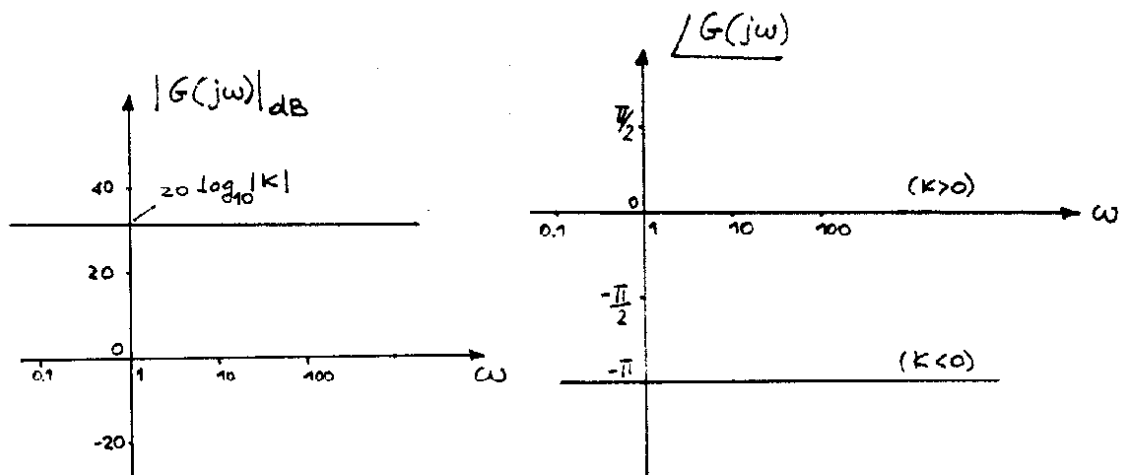
DIAGRAMMI DI BODE: PARAMETRI CARATTERISTICI



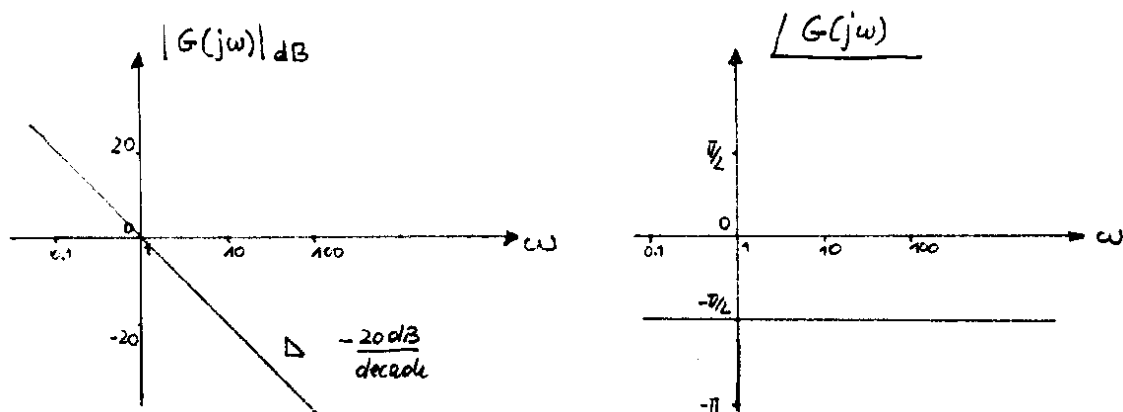
- Picco di risonanza: M_r
- Pulsazione di risonanza: ω_r
- Banda a 3dB: B_3
- Pulsazione di attraversamento: ω_a

DIAGRAMMI DI BODE DI SISTEMI ELEMENTARI

- Sistema statico: $G(s) = K$

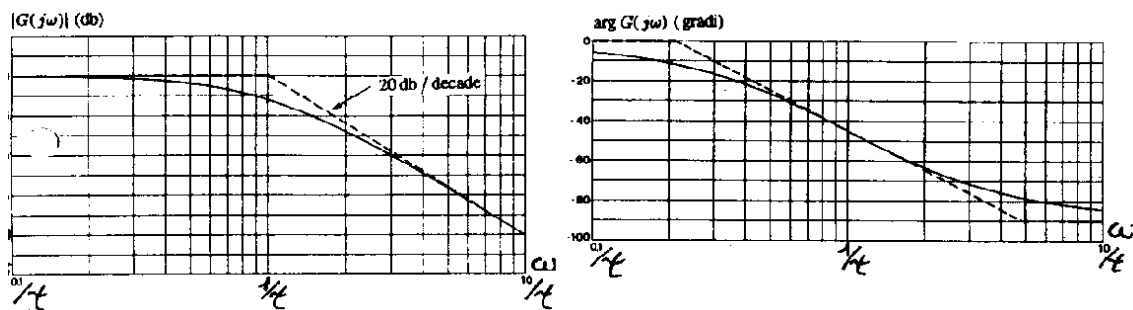


- Integratore: $G(s) = s^{-1}$

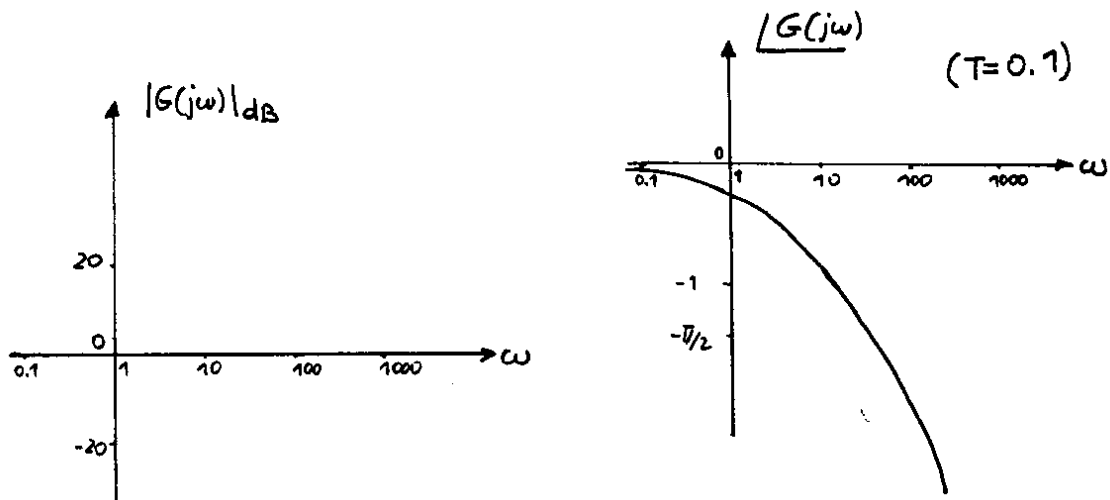


DIAGRAMMI DI BODE DI SISTEMI ELEMENTARI

- Sistema del primo ordine: $G(s) = (1 + \tau s)^{-1}$, ($\tau > 0$).

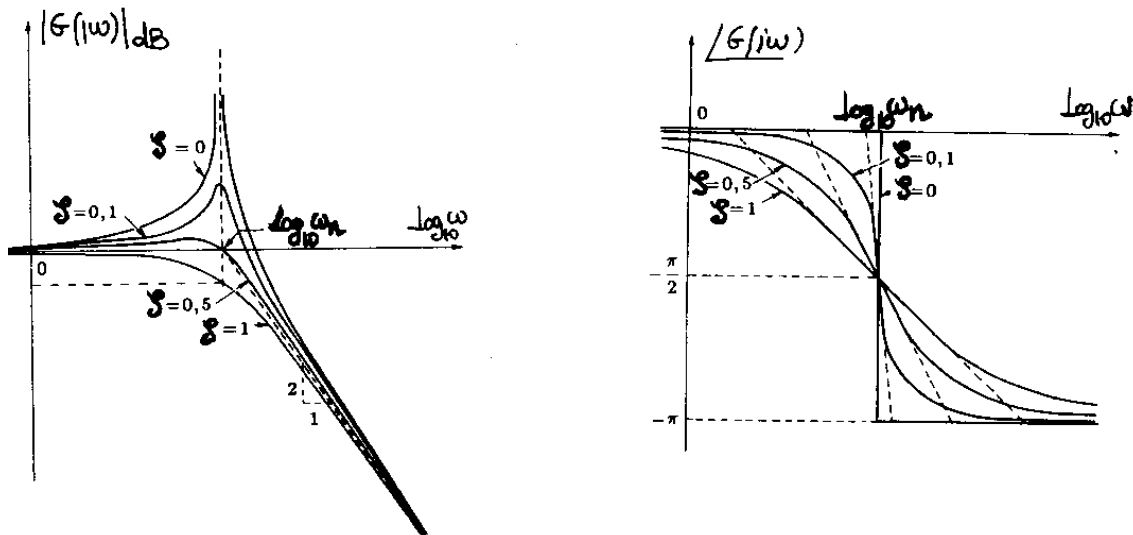


- Elemento di ritardo: $G(s) = e^{-sT}$, ($T > 0$).



DIAGRAMMI DI BODE DI SISTEMI ELEMENTARI

- Sistema del secondo ordine: $G(s) = (1 + 2\zeta \frac{s}{\omega_n} + \frac{s^2}{\omega_n^2})^{-1}$, ($\omega_n > 0; 0 \leq \zeta < 1$).



- Banda a 3dB

$$B_3 = \omega_n \sqrt{1 - 2\zeta^2 + \sqrt{2 - 4\zeta^2 + 4\zeta^4}}$$

- Pulsazione di risonanza

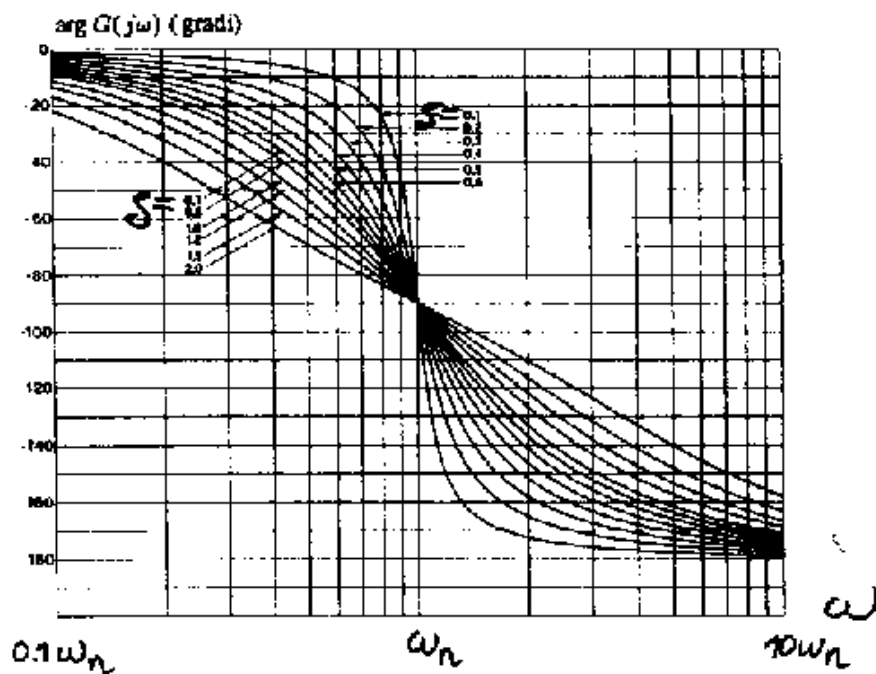
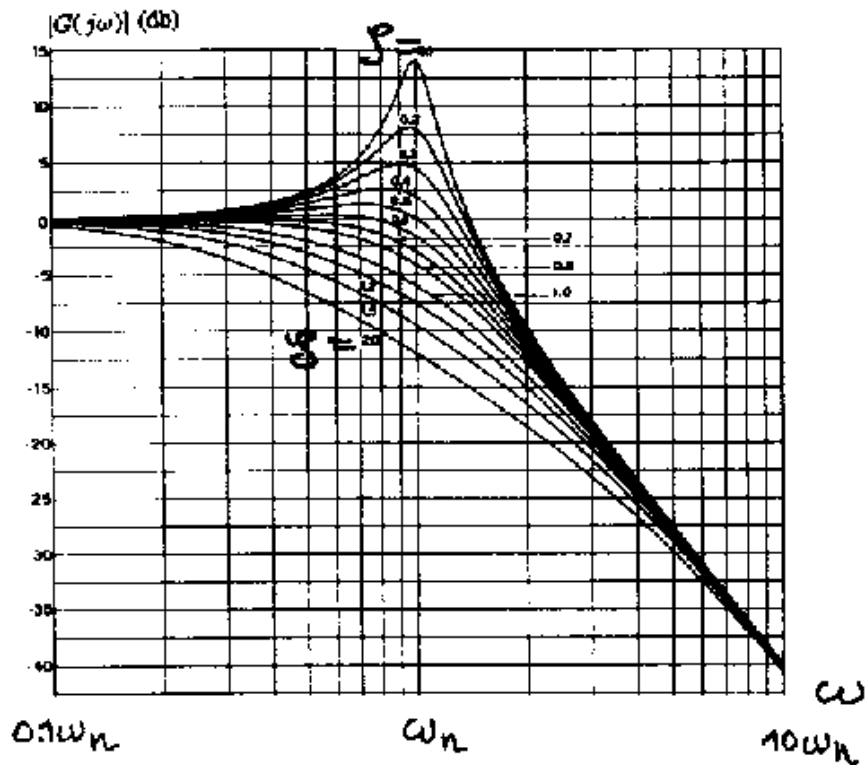
$$\omega_r = \omega_n \sqrt{1 - 2\zeta^2}$$

- Picco di risonanza

$$M_r = \frac{1}{2\zeta \sqrt{1 - \zeta^2}}$$

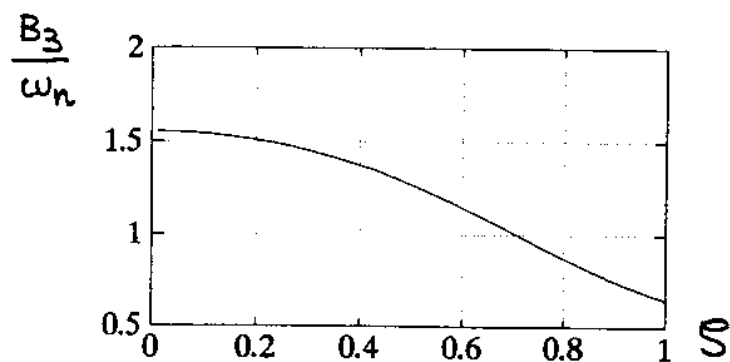
DIAGRAMMI DI BODE DI SISTEMI ELEMENTARI

- Sistema del secondo ordine: $G(s) = (1 + 2\zeta \frac{s}{\omega_n} + \frac{s^2}{\omega_n^2})^{-1}$, ($\omega_n > 0; 0 \leq \zeta < 1$).

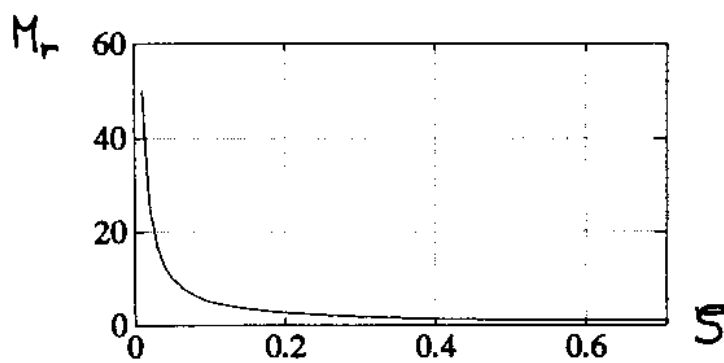


RELAZIONI PARAMETRI SISTEMI SECONDO ORDINE

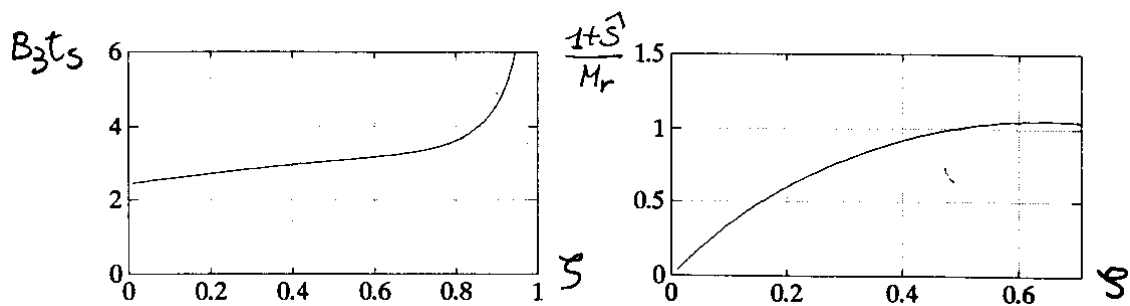
- Banda a 3dB in funzione di ζ



- Modulo alla risonanza in funzione di ζ

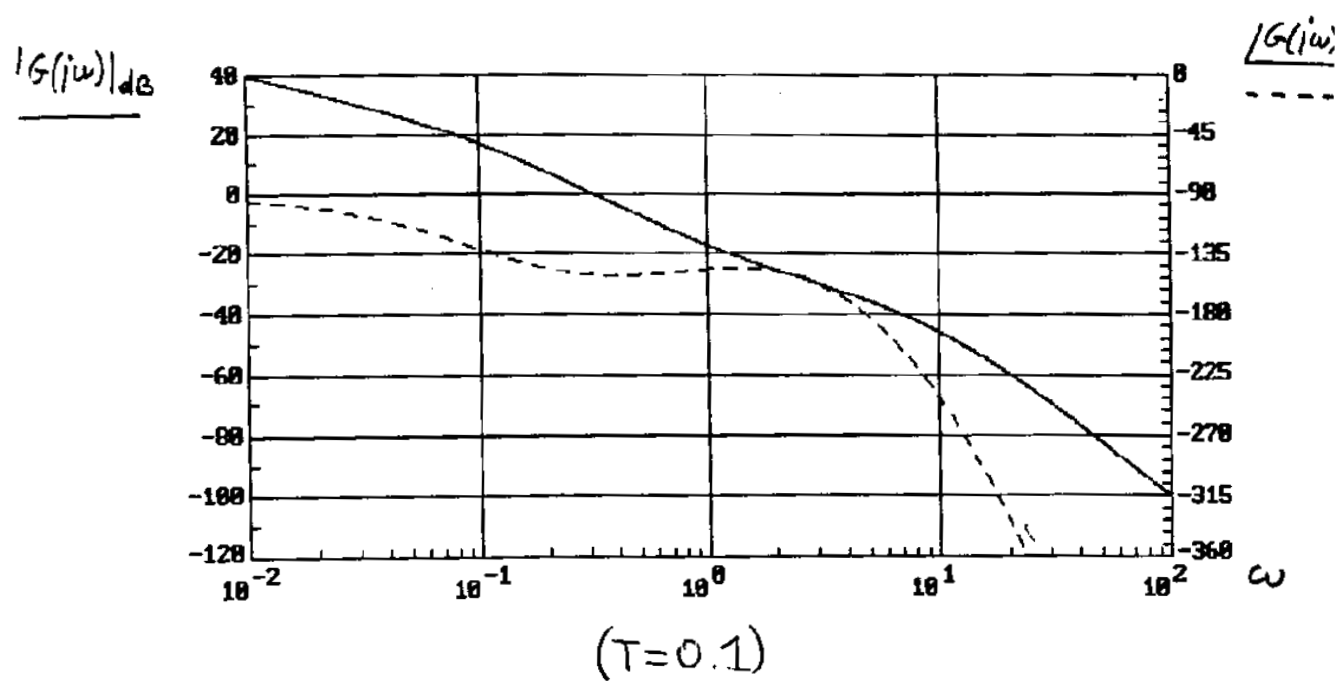
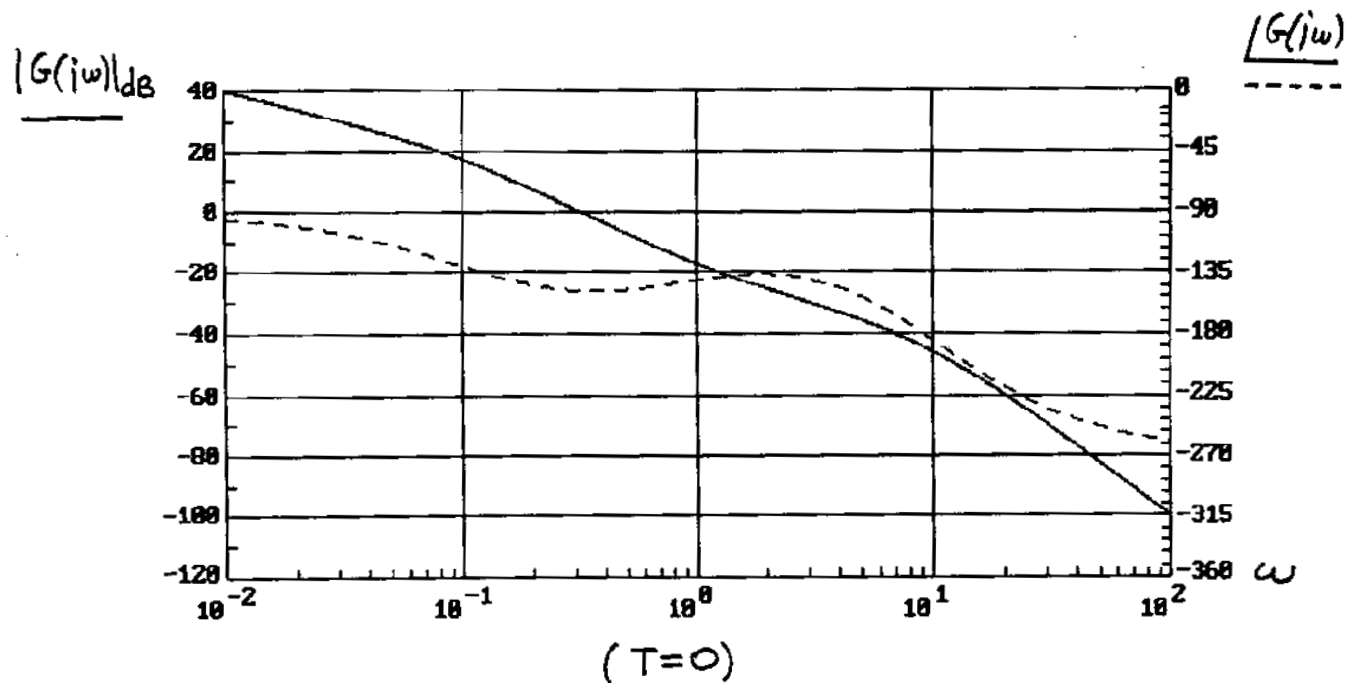


- Relazioni fra \hat{s} e M_r e fra t_s e B_3 .



ESEMPI DI TRACCIAMENTO DIAGRAMMI DI BODE

$$G(s) = e^{-sT} \frac{1+s}{s(1+10s)(1+0.1s)^2}$$



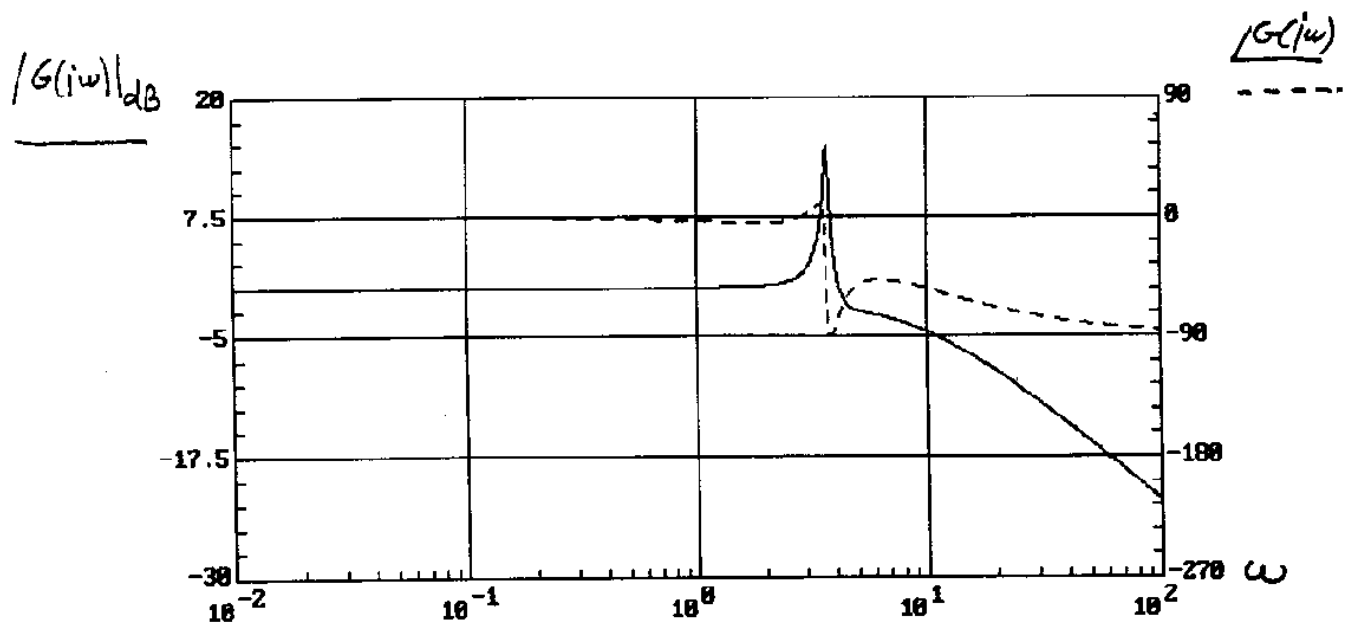
ESEMPI DI TRACCIAMENTO DIAGRAMMI DI BODE

$$G(s) = \frac{8(s^2 + s + 15)}{s^3 + 9s^2 + 15s + 120}$$

- Forma di Bode

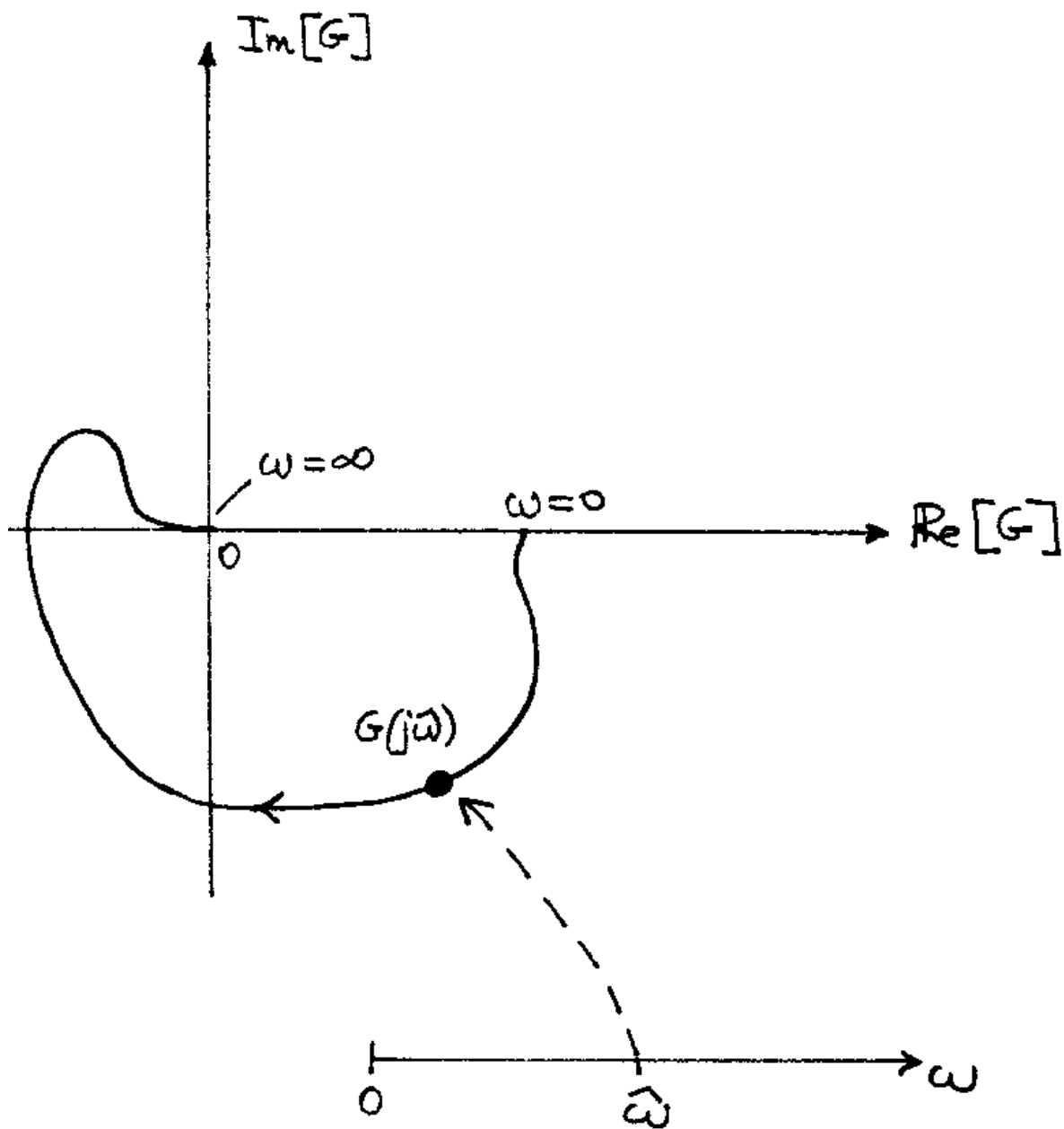
$$G(s) = \frac{1 + 2\zeta_z \frac{s}{\omega_{nz}} + \frac{s^2}{\omega_{nz}^2}}{(1 + s\tau_p)(1 + 2\zeta_p \frac{s}{\omega_{np}} + \frac{s^2}{\omega_{np}^2})}$$

$$\omega_{nz} \approx 3.873; \quad \zeta_z \approx 0.129; \quad \tau_p \approx 0.113; \quad \omega_{np} \approx 3.685; \quad \zeta_p \approx 0.022$$



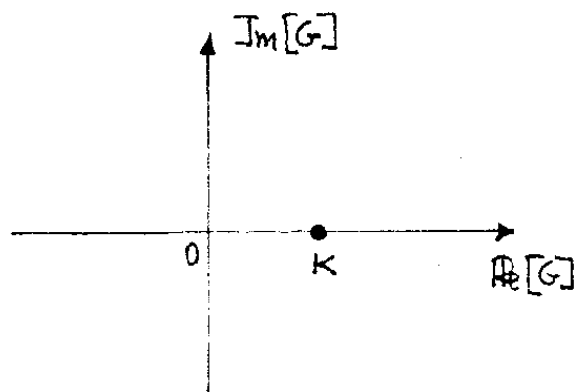
DIAGRAMMI POLARI O DI NYQUIST DI $G(j\omega)$

- Il diagramma polare è la curva nel piano di complesso descritta da $G(j\omega)$ al variare della pulsazione ω in $[0, \infty]$.

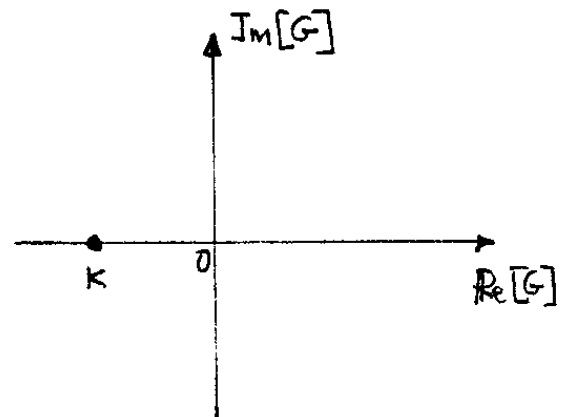


DIAGRAMMI POLARI DI SISTEMI ELEMENTARI

- Sistema statico: $G(s) = K$

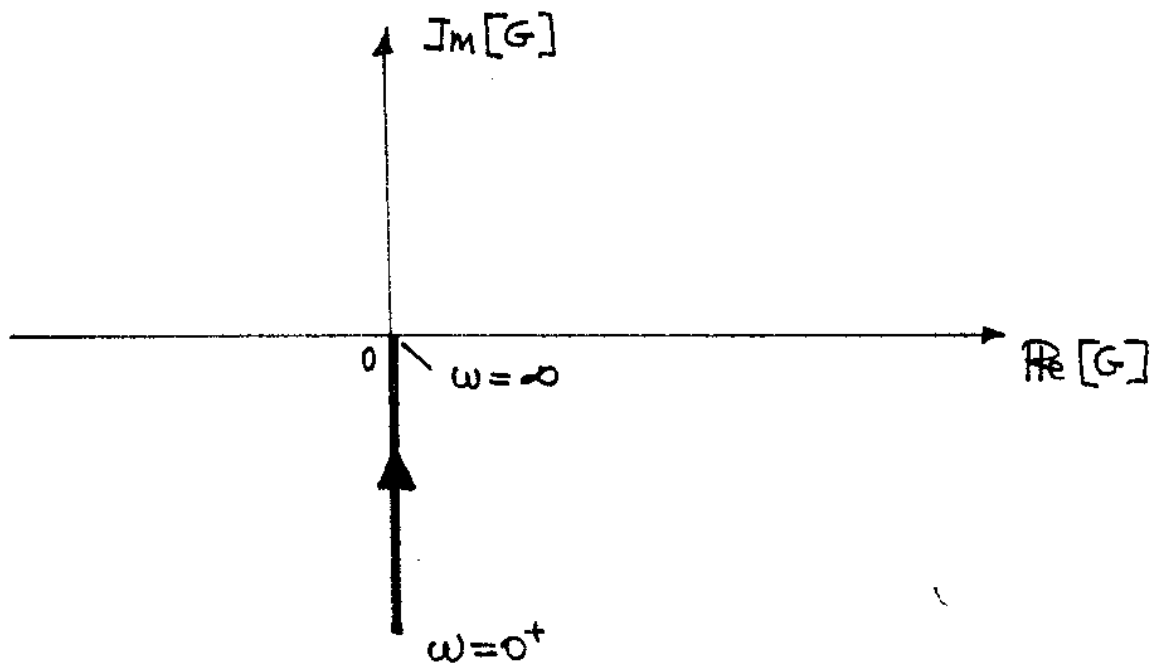


($K > 0$)



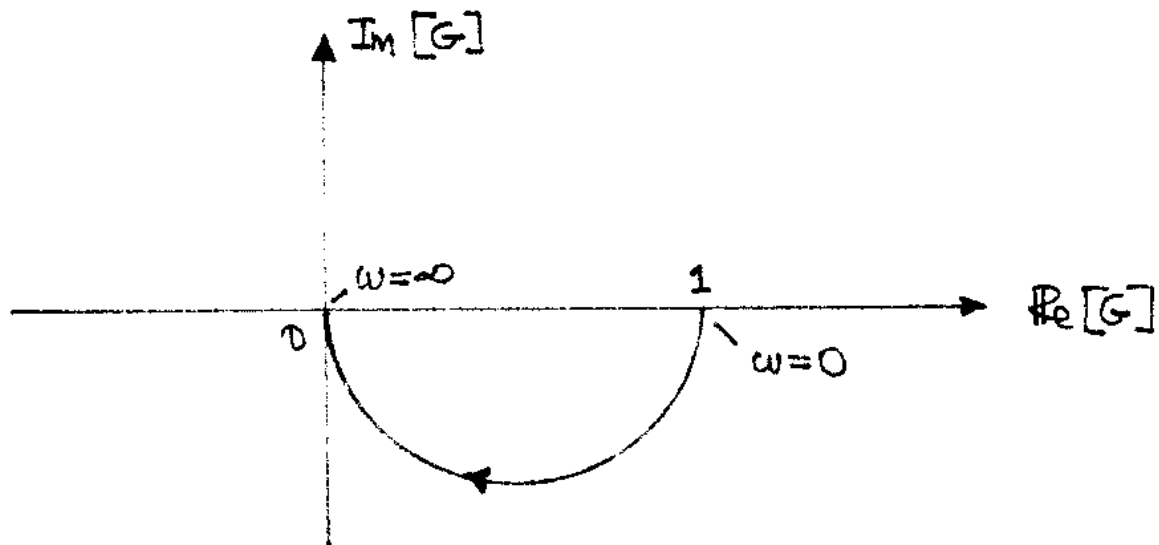
($K < 0$)

- Integratore: $G(s) = s^{-1}$

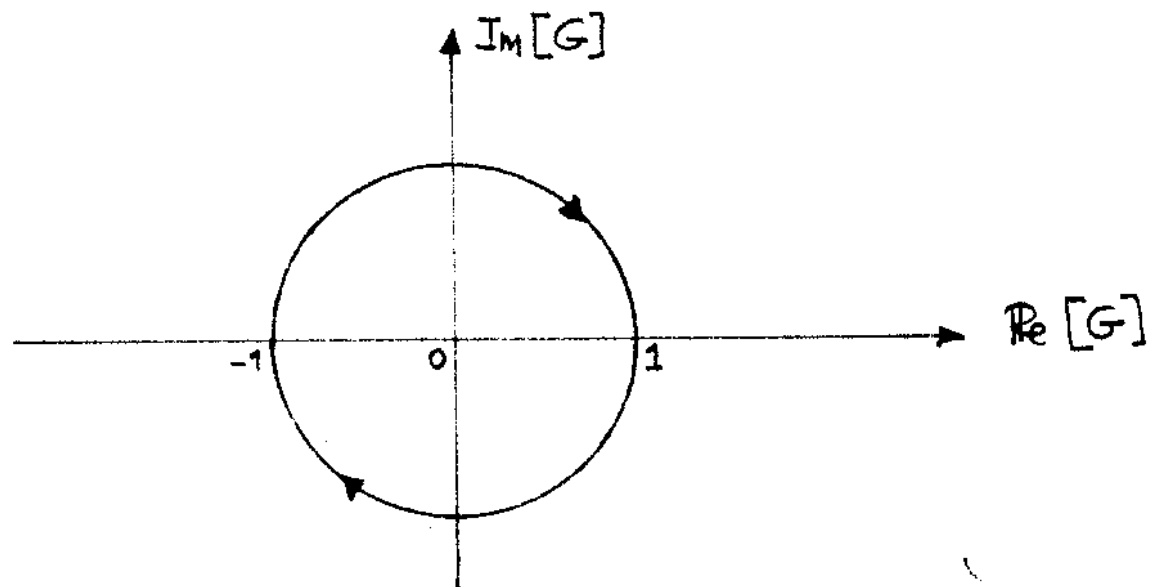


DIAGRAMMI POLARI DI SISTEMI ELEMENTARI

- Sistema del primo ordine: $G(s) = (1 + \tau s)^{-1}$, ($\tau > 0$).

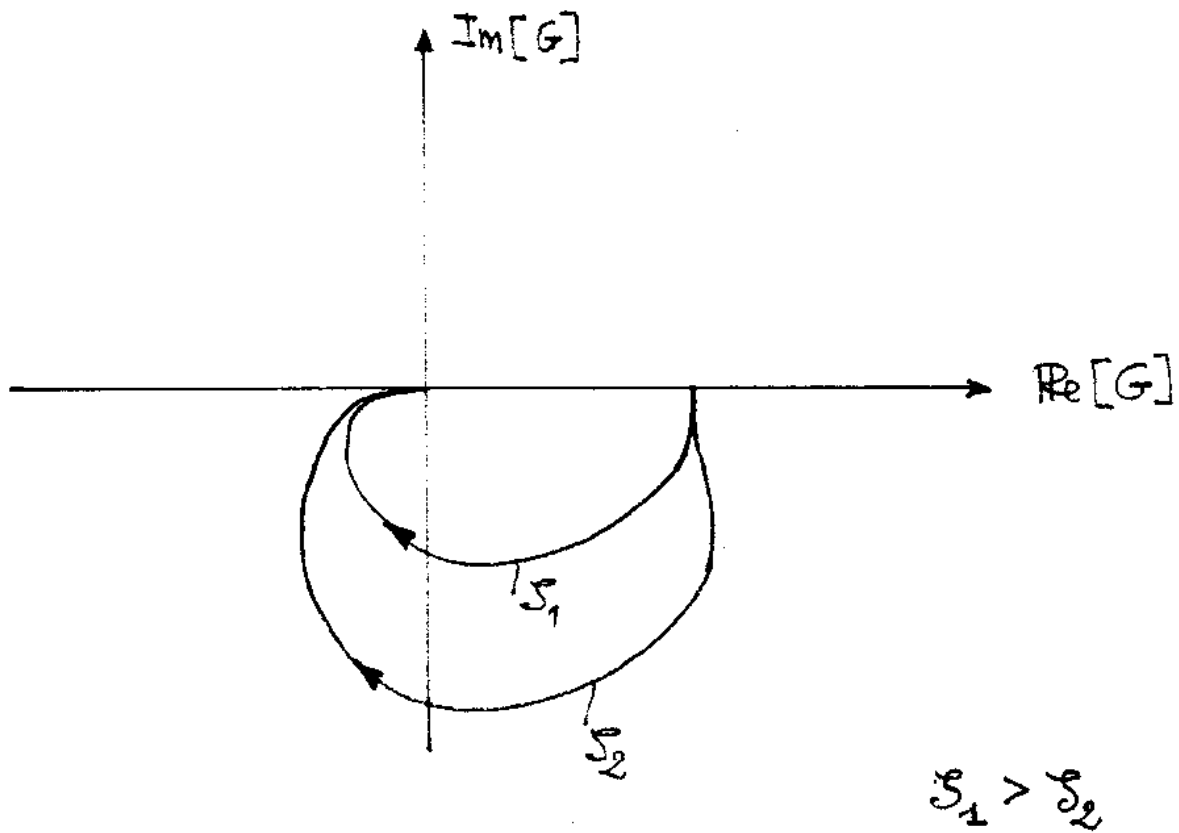


- Elemento di ritardo: $G(s) = e^{-sT}$, ($T > 0$).



DIAGRAMMI POLARI DI SISTEMI ELEMENTARI

- Sistema del secondo ordine: $G(s) = (1 + 2\zeta \frac{s}{\omega_n} + \frac{s^2}{\omega_n^2})^{-1}$, ($\omega_n > 0; 0 \leq \zeta < 1$).



SINGOLARITÀ DEI DIAGRAMMI POLARI

$$G(s) = \frac{1}{s^h} G'(s)$$

- Un polo in zero ($h = 1$) \rightarrow asintoto verticale.

$$G(s) = \frac{G'_o}{s} + G'_1 + G'_2 s + G'_3 s^2 + \dots$$

$$G'_o \doteq G'(0); \quad G'_1 \doteq \frac{d}{ds} G'(s)|_{s=0}; \quad G'_2 \doteq \frac{1}{2!} \frac{d^2}{ds^2} G'(s)|_{s=0}; \quad G'_3 \doteq \frac{1}{3!} \frac{d^3}{ds^3} G'(s)|_{s=0}$$

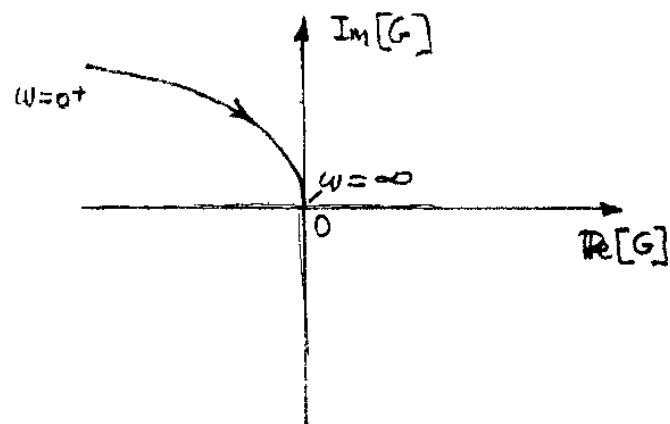
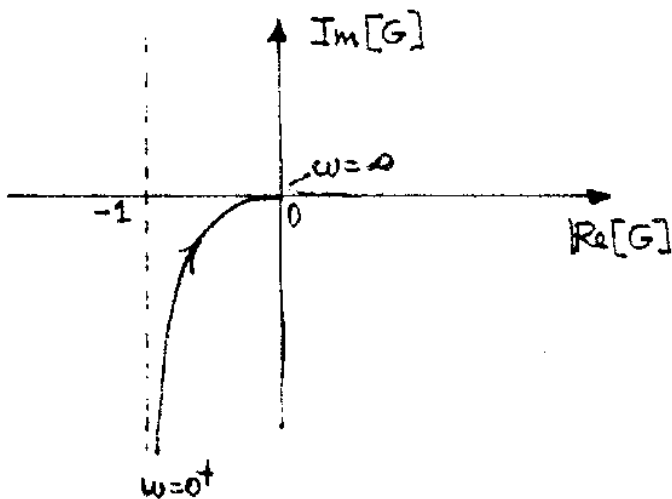
- Un polo doppio in zero ($h = 2$) \rightarrow genericamente asintoto parabolico.

$$G(s) = \frac{G'_o}{s^2} + \frac{G'_1}{s} + G'_2 + G'_3 s \dots$$

- Esempi.

$$G(s) = \frac{1}{s(s+1)}$$

$$G(s) = \frac{1}{s^2(s+1)}$$



SINGOLARITÀ DEI DIAGRAMMI POLARI

- Un polo per $s = j\hat{\omega} \rightarrow$ asintoto obliquo.

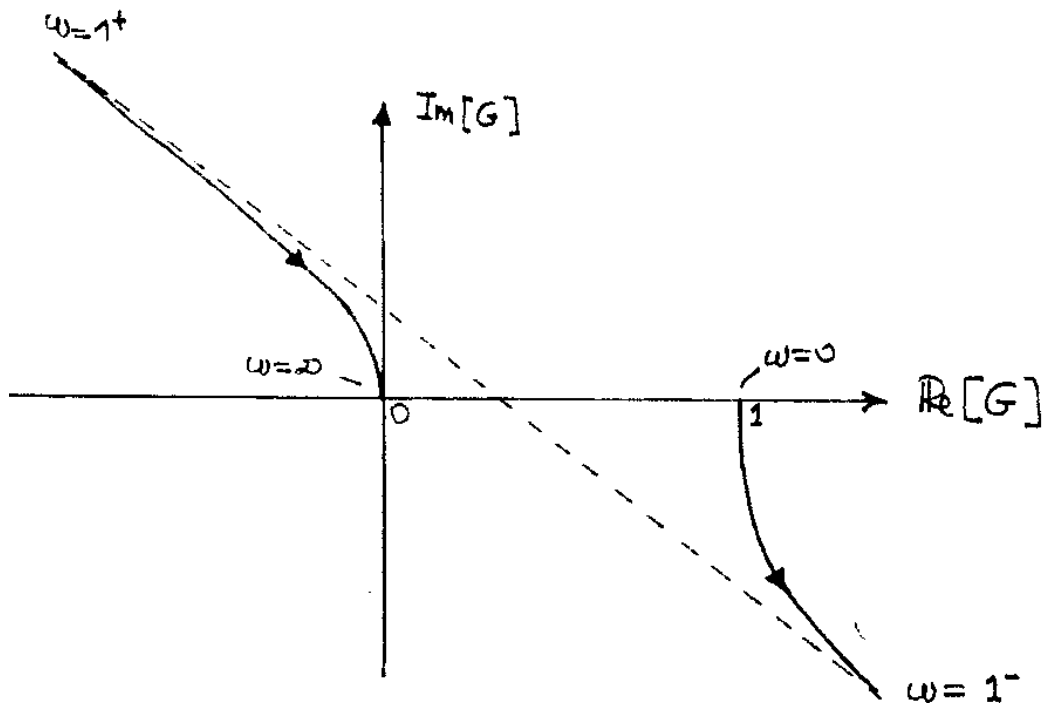
$$G(s) = \frac{1}{s^2 + \hat{\omega}^2} G'(s) = \frac{1}{s - j\hat{\omega}^2} G''(s)$$

$$G(s) = \frac{G''_0}{s - j\hat{\omega}} + G''_1 + G''_2(s - j\hat{\omega}) + \dots$$

$$G''_0 \doteq G''(j\hat{\omega}); \quad G''_1 \doteq \frac{d}{ds} G''(s)|_{s=j\hat{\omega}}; \quad G''_2 \doteq \frac{1}{2!} \frac{d^2}{ds^2} G''(s)|_{s=j\hat{\omega}}$$

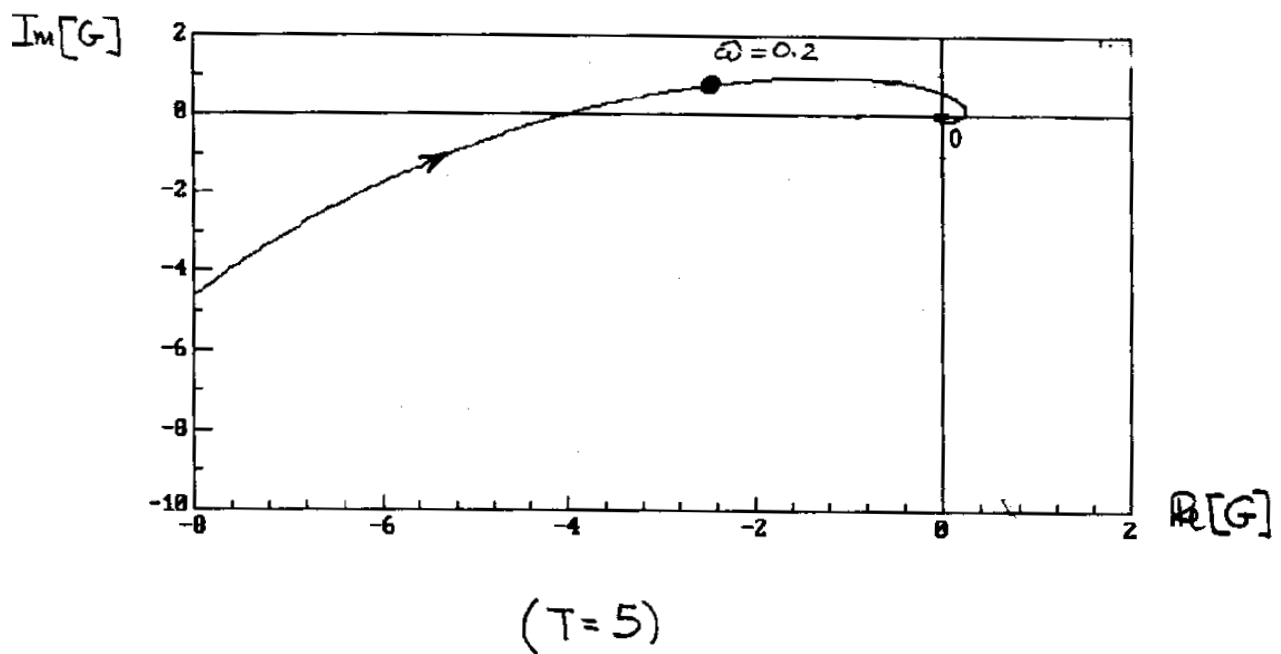
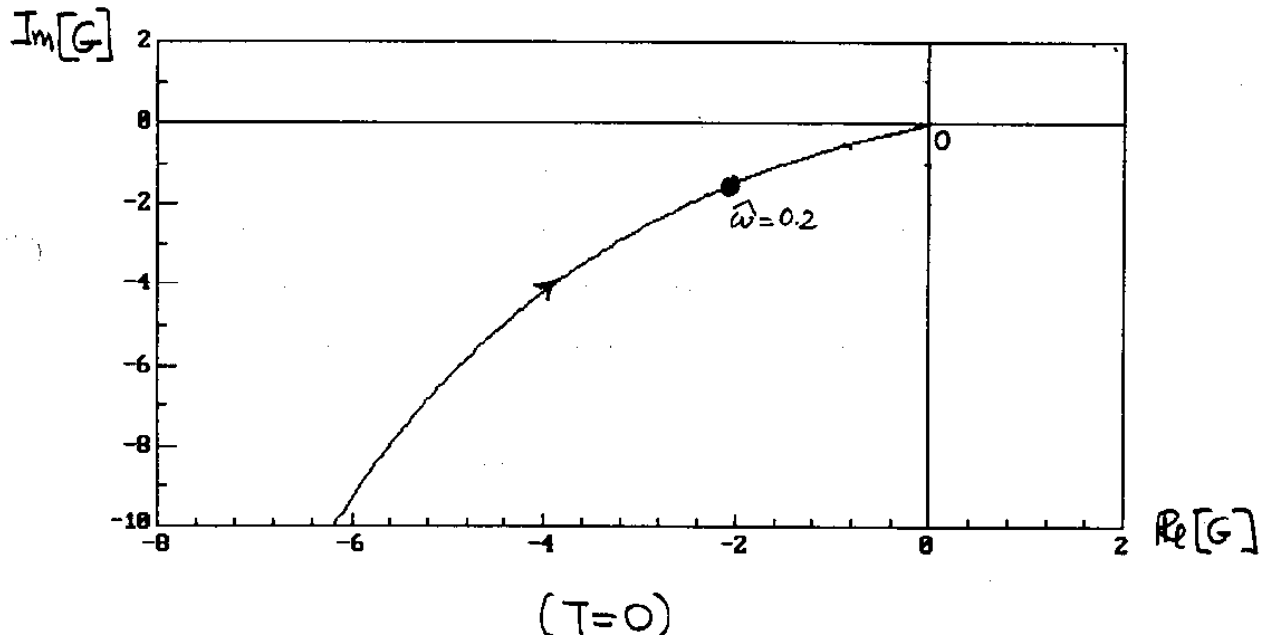
- Esempio.

$$G(s) = \frac{1}{(1+s)(1+s^2)}$$



ESEMPI DI TRACCIAMENTO DIAGRAMMI POLARI

$$G(s) = e^{-sT} \frac{1+s}{s(1+10s)(1+0.1s)^2}$$



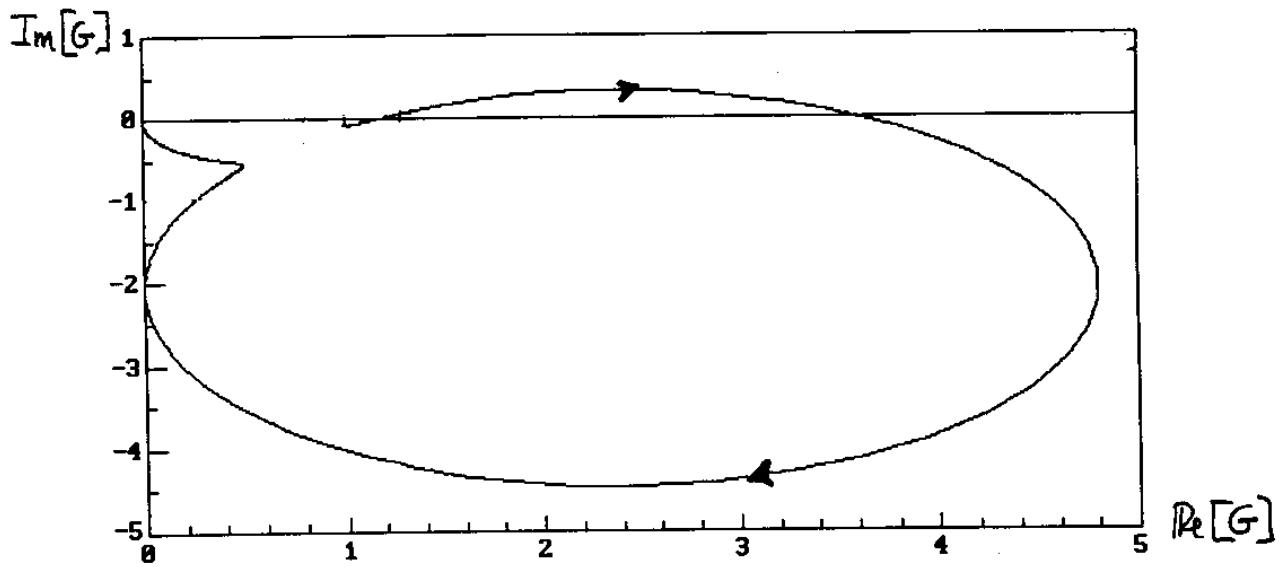
ESEMPI DI TRACCIAMENTO DIAGRAMMI POLARI

$$G(s) = \frac{8(s^2 + s + 15)}{s^3 + 9s^2 + 15s + 120}$$

- Forma di Bode

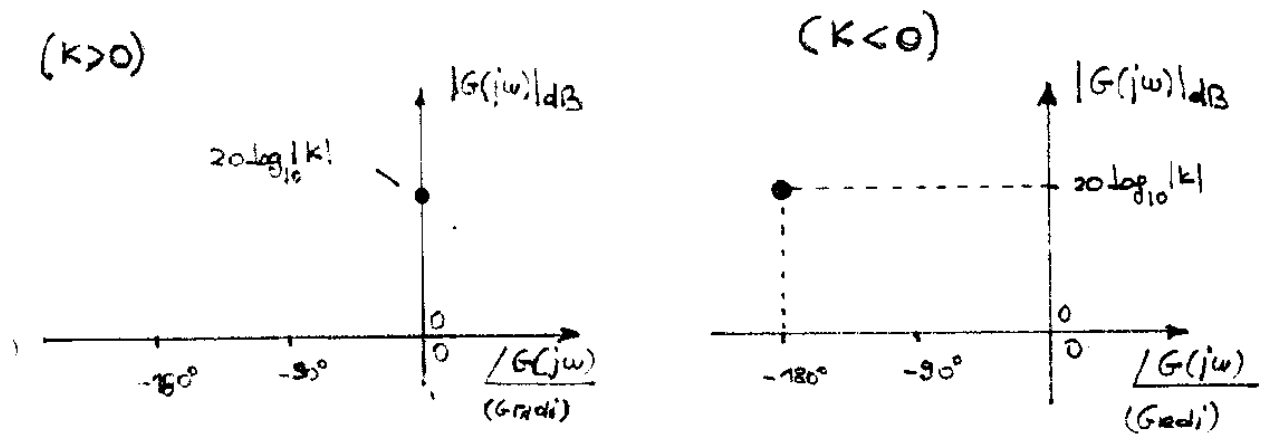
$$G(s) = \frac{1 + 2\zeta_z \frac{s}{\omega_{nz}} + \frac{s^2}{\omega_{nz}^2}}{(1 + s\tau_p)(1 + 2\zeta_p \frac{s}{\omega_{np}} + \frac{s^2}{\omega_{np}^2})}$$

$$\omega_{nz} \approx 3.873; \quad \zeta_z \approx 0.129; \quad \tau_p \approx 0.113; \quad \omega_{np} \approx 3.685; \quad \zeta_p \approx 0.022$$

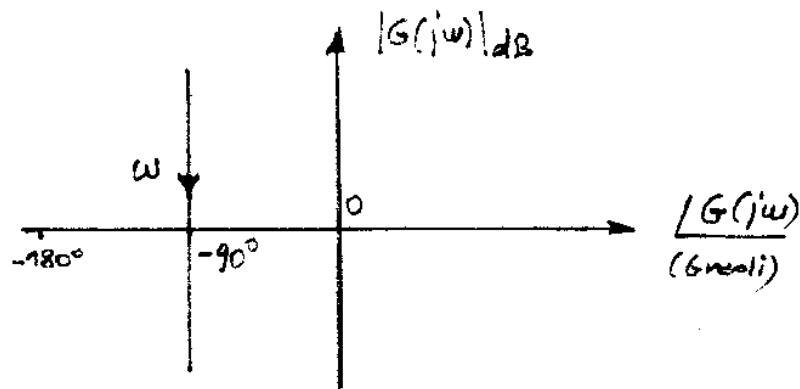


DIAGRAMMI DI NICHOLS DI SISTEMI ELEMENTARI

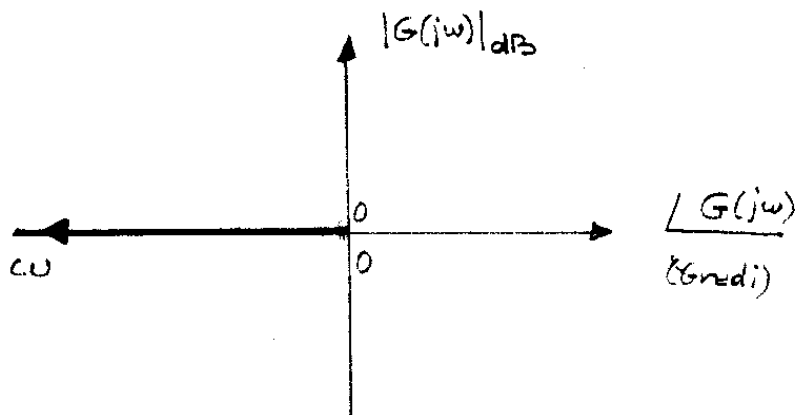
- Sistema statico: $G(s) = K$



- Integratore: $G(s) = s^{-1}$

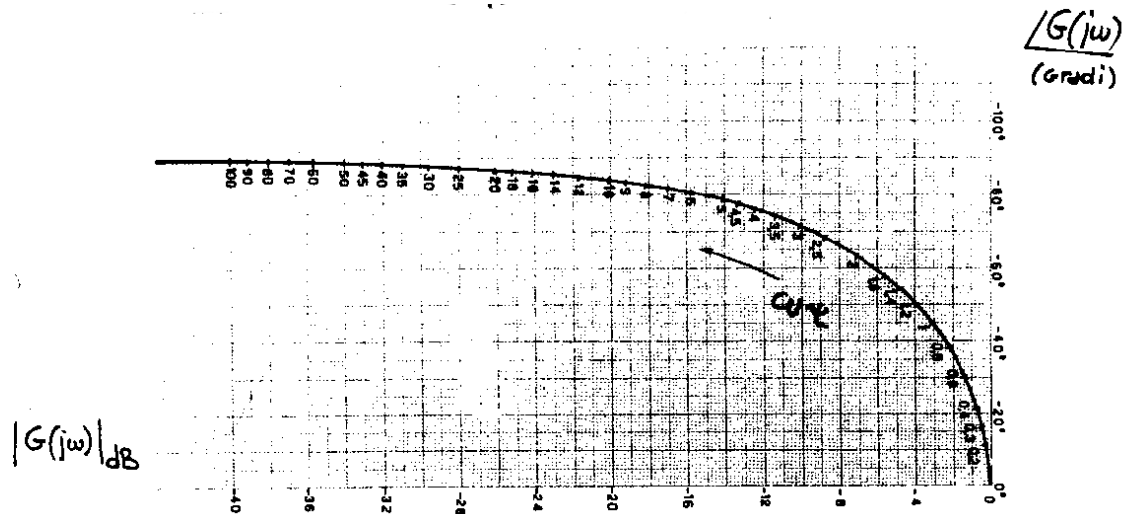


- Elemento di ritardo: $G(s) = e^{-sT}$

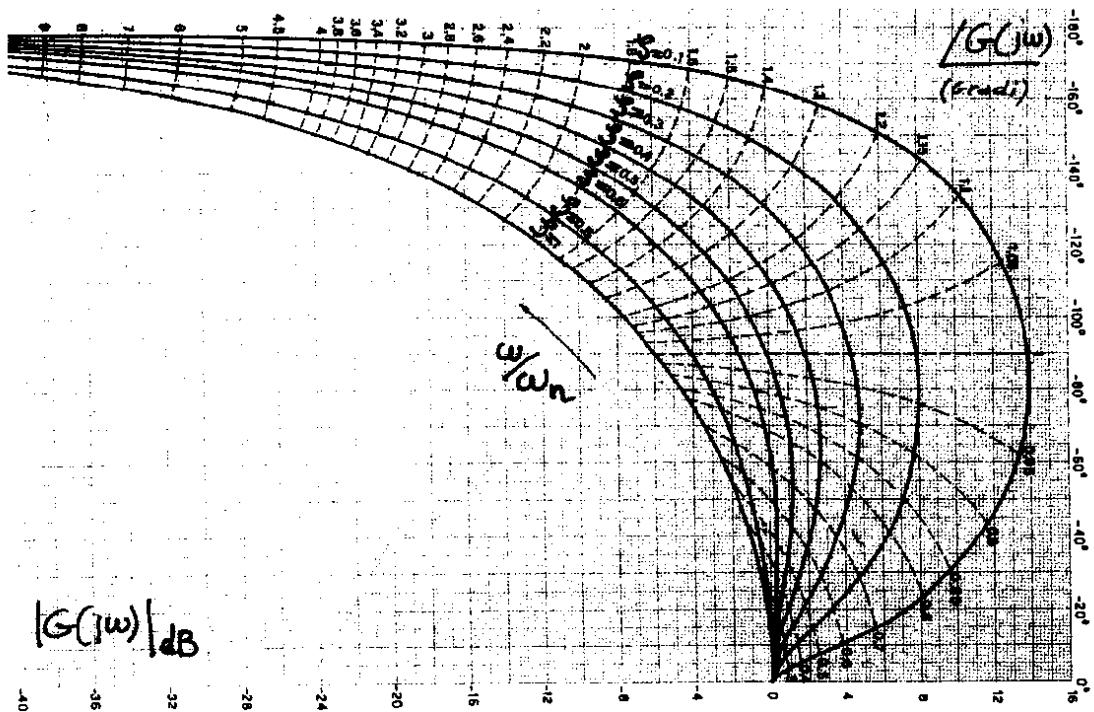


DIAGRAMMI DI NICHOLS DI SISTEMI ELEMENTARI

- Sistema del primo ordine: $G(s) = (1 + \tau s)^{-1}$, ($\tau > 0$).



- Sistema del secondo ordine: $G(s) = (1 + 2\zeta \frac{s}{\omega_n} + \frac{s^2}{\omega_n^2})^{-1}$, ($\omega_n > 0; 0 \leq \zeta < 1$).



RELAZIONI PARTE REALE E IMMAGINARIA DI $G(j\omega)$

- Ipotesi su $G(s)$
 - (I) Analitica nel semipiano destro chiuso del piano complesso
 - (II) Propria ($\lim_{s \rightarrow \infty} G(s) = G_\infty$)
 - (III) $G(s^*) = G^*(s)$ (* sta per coniugato)
- Relazione fra $R(j\omega) := \text{Re}[G(j\omega)]$ e $I(j\omega) := \text{Im}[G(j\omega)]$ (coppia di trasformate di Hilbert)

$$R(j\omega_o) - G_\infty = -\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{I(j\omega)}{\omega - \omega_o} d\omega =$$

$$I(j\omega_o) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{R(j\omega)}{\omega - \omega_o} d\omega =$$

TEOREMA DI BODE

- Parte reale e parte immaginaria di $W(s) := \ln G(s)$

$$\operatorname{Re}[W(j\omega)] = \ln|G(j\omega)|$$

$$\operatorname{Im}[W(j\omega)] = \arg G(j\omega)$$

- Ipotesi: $G(s)$ razionale fratta, propria, con coefficienti reali, con poli e zeri a parte reale strettamente minore di zero (fase minima)

- Teorema di Bode

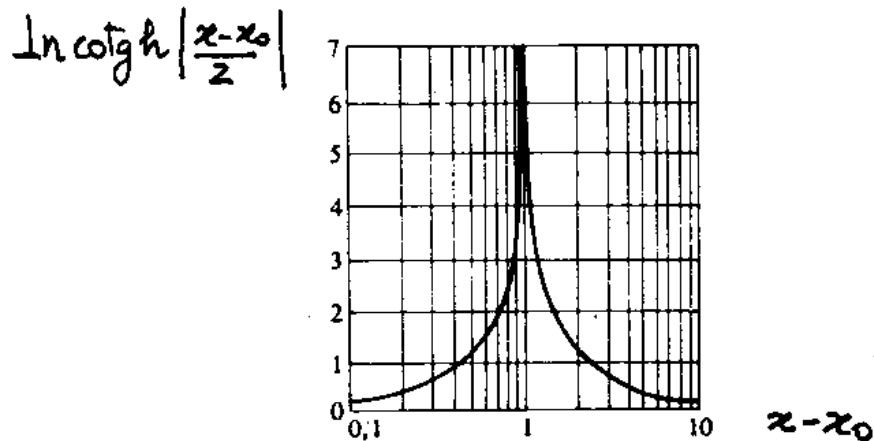
– Posizioni

$$x := \ln \omega; \quad y(x) := \ln|G(j\omega)|; \quad z(x) := \arg G(j\omega)$$

– Formula (modulo \rightarrow fase)

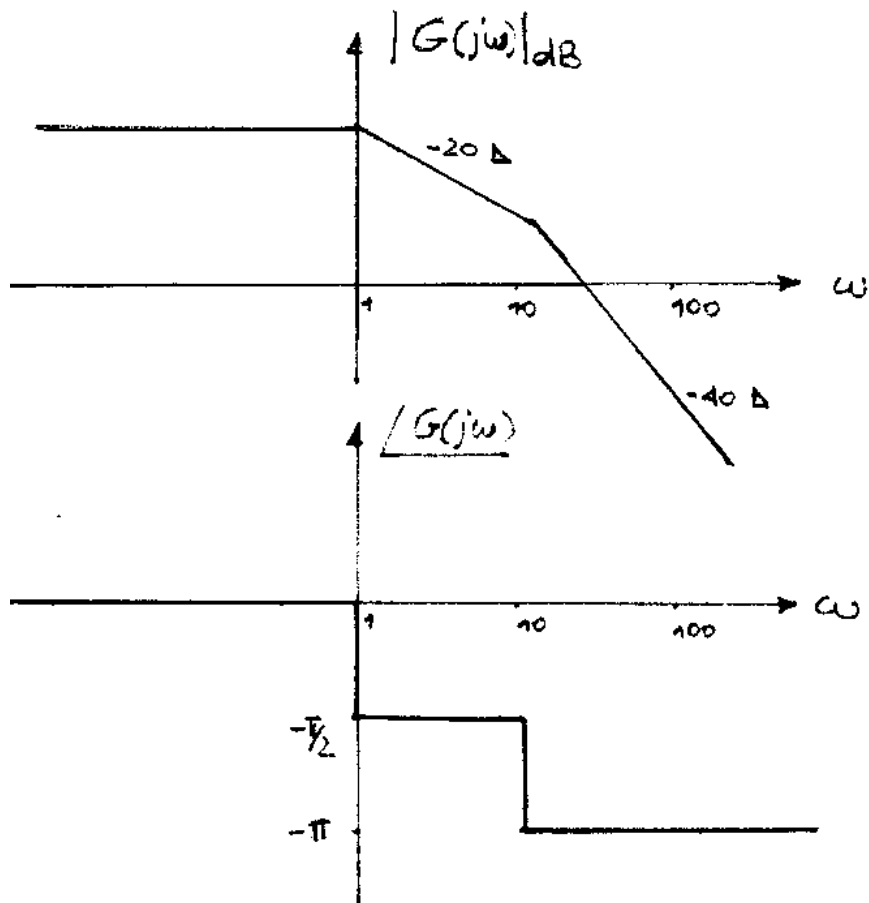
$$z(x_o) = \frac{\pi}{2} \cdot \frac{dy}{dx} \Big|_{x=x_o} + \frac{1}{\pi} \int_{-\infty}^{+\infty} \left\{ \frac{dy}{dx} - \frac{dy}{dx} \Big|_{x=x_o} \right\} \operatorname{Incotgh} \left| \frac{x - x_o}{2} \right| dx$$

– Interpretazione formula



APPLICAZIONE TEOREMA DI BODE

- $G(s)$ è a fase minima



- $G(s)$ non a fase minima si può scrivere

$$G(s) = G_m(s)G_u(s), \quad G_m(s) \text{ a fase minima}, \quad |G_u(j\omega)| = 1$$

– Esempi

$$G_1(s) = \frac{1-s}{(1+2s)^2} = \frac{1+s}{(1+2s)^2} \frac{1-s}{1+s}$$

$$G_2(s) = \frac{1}{1-s+s^2} = \frac{1}{1+s+s^2} \frac{1+s+s^2}{1-s+s^2}$$