In the nonlinear state estimation problem, the objective is to find

$$f_X(x(t) \mid Y^t)$$

(recall that the mean of the above pdf is the MSE estimate)

$$\hat{X}_{MSE}(t) = E[x(t) \mid Y^t] =$$
$$= \int x \cdot f_X(x \mid Y^t)\, dx$$

In the linear Gaussian case, the KF gives

$$f_X(x(t) \mid Y^t) = N(\hat{x}(t|t), P(t|t))$$

In the nonlinear case, the EKF approximates $f_X(x(t) \mid Y^t)$ with the Gaussian distribution $N(\hat{x}(t|t), P(t|t))$.

When the approximation of the posterior pdf $f_X(x(t) \mid Y^t)$ provided by the EKF is not "good enough", we must try different methods to approximate the posterior pdf.

Examples:

* the Unscented Kalman Filter (UKF)
* the Particle Filter (PF)

THE UNSCENTED KALMAN FILTER

Basic idea: propagate a set of points in the space of $x(t)$ which are representative of some statistics of $f_x(x(t) \mid Y^t)$.

Sigma points $\quad X^{(i)}$

Weights $\quad W^{(i)} \qquad i = 1 \dots p$

such that $X^{(i)} \in \mathbb{R}^n$ and $\sum_{i=1}^{p} W^{(i)} = 1$

Example: match mean and covariance of a n-variate Gaussian $N(m, P)$

Need $p = 2n$

$$X^{(i)} = m + \left(\sqrt{nP}\right)_i \qquad i = 1 \dots, n$$

$$X^{(i)} = m - \left(\sqrt{nP}\right)_i \qquad i = n+1, \dots, 2n$$

$$W^{(i)} = \frac{1}{2n} \qquad i = 1, \dots, 2n$$

where if M is a matrix, $\sqrt{M} = Q$ so that $Q \cdot Q^T = M$ and $(M)_i$ is the i-th column of M

It is easy to show that the sample mean of $x^{(i)}$ is m and their sample variance is P, i.e.

$$\sum_{i=1}^{2n} w^{(i)} x^{(i)} = m$$

$$\sum_{i=1}^{2n} w^{(i)} \left(x^{(i)} - m\right)\left(x^{(i)} - m\right)^T = P$$

Given a nonlinear transformation

$$z = h(x)$$

compute the transformed Sigma Points

$$z^{(i)} = h\left(x^{(i)}\right) \qquad i = 1, \dots, p$$

then, approximate mean and covariance of z by

$$\bar{z} = \sum_{i=1}^{p} w^{(i)} z^{(i)}$$

$$\mathrm{Cov}(z) = \sum_{i=1}^{p} w^{(i)} \left(z^{(i)} - \bar{z}\right)\left(z^{(i)} - \bar{z}\right)^T$$

Application of the "unscented transform" to the state estimation problem.

UKF algorithm.

0) Let $\hat{x}(t|t)$ and $P(t|t)$ be known.

1) Generate the sigma points $\{x^{(i)}, w^{(i)}\}$ from the distribution $N(\hat{x}(t|t), P(t|t))$ (or other distribution with mean $\hat{x}(t|t)$ and covariance $P(t|t)$)

2) Prediction

$$\hat{x}^{(i)} = f(x^{(i)}, u(t), \xi^{(i)})$$

where the $\xi^{(i)}$ are sigma points from the distribution of the disturbance process $w(t)$.

$$\hat{x}(t+1|t) = \sum_{i=1}^{P} w^{(i)} \hat{x}^{(i)}$$

$$P(t+1|t) = \sum_{i=1}^{P} w^{(i)} \{\hat{x}^{(i)} - \hat{x}(t+1|t)\} \cdot \{\hat{x}^{(i)} - \hat{x}(t+1|t)\}^T$$

3) Correction

$$\hat{y}^{(i)} = h(\hat{x}^{(i)}) + v^{(i)}$$

where the $v^{(i)}$ are sigma points from the distribution of the measurement noise $v(t)$

Define

$$\hat{y}(t+1) = \sum_{i=1}^{P} w^{(i)} \hat{y}^{(i)}$$

$$S(t+1) = \sum_{i=1}^{P} w^{(i)} \{\hat{y}^{(i)} - \hat{y}(t+1)\}\{\hat{y}^{(i)} - \hat{y}(t+1)\}^T$$

$$P_{XY}(t+1) = \sum_{i=1}^{P} w^{(i)} \{\hat{x}^{(i)} - \hat{x}(t+1|t)\}\{\hat{y}^{(i)} - \hat{y}(t+1)\}^T$$

$$\hat{x}(t+1|t+1) = \hat{x}(t+1|t) + P_{XY}(t+1) \cdot S(t+1)^{-1} \cdot \{y(t+1) - \hat{y}(t+1)\}$$

$$P(t+1|t+1) = P(t+1|t) - P_{XY}(t+1)\, S(t+1)^{-1} P_{XY}^T(t+1)$$

SEQUENTIAL MONTE CARLO ESTIMATORS

Key idea: approximate the posterior pdf $f_x(x(t) | Y^t)$ through a (large) number of points ("particles")

The Particle Filter

0) The points (particles) x^i_{t-1}, $i=1,..,M$ are distributed according to the pdf $f_x(x(t-1), Y^{t-1})$

1) Prediction

$$\hat{x}^i_t = f(x^i_{t-1}, u(t), w^i_{t-1}) \quad i=1,..,M$$

where the w^i_{t-1} are particles generated from the pdf of $w(t)$

Now, the particles $\hat{x}^i_t$ are distributed according to $f_x(x(t) | Y^{t-1})$

2) Correction

Generate the weights

$$q_i = \frac{f_Y(y(t) \mid \hat{x}_t^i)}{\sum_{j=1}^{M} f_Y(y(t) \mid \hat{x}_t^j)} \qquad i=1,..,M$$

$q_i \in [0,1]$ and $\sum_{i=1}^{M} q_i = 1$

Re-sample M times from the set of particles $\hat{x}_t^i$, so that the probability of extracting $\hat{x}_t^i$ is q_i.

Generate x_t^j such that

$$\mathbb{P}\{x_t^j = \hat{x}_t^i\} = q_i \quad \text{for } j=1,..,M$$

The particles x_t^j, $j=1\ldots M$ are distributed according to $f_X(x(t) \mid Y^t)$

... we can iterate!

If we need an estimate of the mean and covariance, we can use

$$\hat{x}(t|t) = \frac{1}{M} \sum_{i=1}^{M} x_t^i$$

$$P(t|t) = \frac{1}{M} \sum_{i=1}^{M} \left(x_t^i - \hat{x}(t|t)\right)\left(x_t^i - \hat{x}(t|t)\right)^T$$

Examples

1) $x(t+1) = x(t) + w(t)$ $\quad w \sim N(0, 0.5)$

$y(t) = x^2(t) + v(t)$ $\quad v \sim N(0, 0.05)$

2) $x(t+1) = x(t)$

$y(t+1) = y(t)$

$d_1(t) = \sqrt{x^2(t) + y^2(t)} + v_1(t)$

$d_2(t) = \sqrt{(x(t)-1)^2 + (y(t)-1)^2} + v_2(t)$

$v_i(t) \sim N(0, 0.04)$ independent