NONLINEAR STATE ESTIMATION

$$x(t+1) = f(x(t), u(t), w(t))$$
$$y(t) = h(x(t)) + v(t)$$

$x(t) \in \mathbb{R}^n$: state vector
$u(t) \in \mathbb{R}^m$: determeministic input (known)
$w(t) \in \mathbb{R}^d$: process disturbance

$w(t) \in WN(0, Q)$

$f(x, u, w): \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}^n$

$y(t) \in \mathbb{R}^p$: output vector
$v(t) \in \mathbb{R}^p$: measurement noise
$v(t) \in WN(0, R)$

$w(t)$, $v(t)$, $x(0)$ are independent

$h(x): \mathbb{R}^n \to \mathbb{R}^p$

Assumption: $f(\cdot, \cdot, \cdot)$ and $h(\cdot) \in C^1$

State estimation problem:

Compute an estimate $\hat{x}(t|t)$ of $x(t)$ based on the knowledge of $u(k)$ $\forall k$ and of $Y^t = \{y(0), y(1), \dots y(t)\}$.

Extended Kalman Filter (EKF)

Prediction step

$$\hat{x}(t+1|t) = f(\hat{x}(t|t), u(t), \emptyset)$$
$$P(t+1|t) = F(t) P(t|t) F^T(t) + G(t) Q G^T(t)$$

Where

$$F(t) \triangleq \left.\frac{\partial f}{\partial x}\right|_{\substack{x=\hat{x}(t|t)\\ u=u(t)\\ w=\emptyset}} \qquad G(t) \triangleq \left.\frac{\partial f}{\partial w}\right|_{\substack{x=\hat{x}(t|t)\\ u=u(t)\\ w=\emptyset}}$$

Correction step

$$\hat{x}(t+1|t+1) = \hat{x}(t+1|t) + K(t+1)\left(y(t+1) - h(\hat{x}(t+1|t)\right.$$

$$K(t+1) = P(t+1|t)\,H^T(t+1)\left[H(t+1)P(t+1|t)\,H^T(t+1) + R\right]^{-1}$$

$$P(t+1|t+1) = P(t+1|t)\left[I - H^T(t+1)K^T(t+1)\right]$$

where

$$H(t+1) = \left.\frac{\partial h}{\partial x}\right|_{x=\hat{x}(t+1|t)}$$

Observations on the EKF

1) The computed estimates $\hat{x}(t|t)$ and $\hat{x}(t+1|t)$ are not guaranteed to be the LMSE estimates of $x(t)$ and $x(t+1)$ based on Y^t

   Moreover, $P(t|t)$ and $P(t+1|t)$ are not the true covariances of $\tilde{x}(t|t)$, $\tilde{x}(t+1|t)$.

2) The matrices $F(t), G(t), H(t)$ depend on the current estimates $\hat{x}(t|t)$, $\hat{x}(t+1|t)$, which in turn depend on the data realization Y^t.

   Therefore the gain and P matrices of the EKF cannot be pre-computed.

3) Possible problems:

   * divergence

   $$\lim_{t \to +\infty} \| x(t) - \hat{x}(t|t) \| = +\infty$$

   or $$\lim_{t \to +\infty} P(t|t) = +\infty$$

* inconsistency

at some time t

$$P(t|t) \ll E\left[\left(x(t)-\hat{x}(t|t)\right)\left(x(t)-\hat{x}(t|t)\right)^T\right]$$

4) Extension : the continuous-discrete EKF

$$\dot{x}(t) = f(x(t), u(t)) + G w(t) \qquad t \in \mathbb{R}$$

$$y(t_k) = h(x(t_k)) + v(t_k) \qquad k \in \mathbb{Z}$$

prediction : for $t \in [t_k, t_{k+1}]$

$$\frac{d}{dt}\hat{x}(t|t_k) = f(\hat{x}(t|t_k), u(t))$$

$$\frac{d}{dt}P(t|t_k) = F(t)P(t) + P(t)F^T(t) + GQG^T$$

where $F(t) = \left.\frac{\partial f}{\partial x}\right|_{\substack{x=\hat{x}(t_k|t_k) \\ u=u(t)}}$

Correction

.. same as in standard EKF !