Stochastic processes

Definition: a sequence of random variables indexed in time

$t \in \mathbb{R} \rightarrow$ continuous-time SPs

$t \in \mathbb{Z} \rightarrow$ discrete-time SPs

$$X(t) : \underset{\substack{\uparrow \\ \text{event} \\ \text{space}}}{\Omega} \times \underset{\substack{\uparrow \\ \text{time} \\ \text{domain}}}{\mathcal{T}} \longrightarrow \mathbb{R}$$

Hereafter: $\mathcal{T} = \mathbb{Z}$

Cdf of $X(t)$

$$F(x_1, x_2, \dots, x_k ; t_1, t_2, \dots, t_k) =$$

$$= \mathbb{P}\{X(t_1) \leq x_1 \ \& \ X(t_2) \leq x_2 \ \& \dots \& \ X(t_k) \leq x_k\}$$

$\forall\ x_1, x_2, \dots, x_k \in \mathbb{R}$

$\forall\ t_1, t_2, \dots, t_k \in \mathbb{Z}$

$\forall\ k \in \mathbb{N}_+$

Pdf of $X(t)$

$$f_X(x_1, x_2, \ldots, x_k; t_1, t_2, \ldots, t_k) =$$

$$= \frac{\partial^k}{\partial x_1 \partial x_2 \ldots \partial x_k} F_X(x_1, x_2, \ldots, x_k; t_1, t_2, \ldots, t_k)$$

Mean and covariance function

The mean of $X(t)$ is defined as

$$m_X(t) = E[X(t)] = \int_{-\infty}^{+\infty} x \cdot f_X(x; t)\, dx$$

The covariance function of $X(t)$ is defined as

$$R_X(t,s) = E\Big[\big(X(t) - m_X(t)\big)\big(X(s) - m_X(s)\big)\Big] =$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \big(x_1 - m_X(t)\big)\big(x_2 - m_X(s)\big) \cdot f_X(x_1, x_2; t, s)\, dx_1\, dx_2$$

Stationary SPs

<u>Def. 1</u> A SP $X(t)$ is stationary in strong sense if

$$F_X(x_1,\dots,x_k;t_1,\dots,t_k)=$$
$$= F_X(x_1,\dots,x_k;t_1+\tau,\dots,t_k+\tau)$$

$\forall \tau \in \mathbb{Z}$
$\forall x_1,\dots,x_n \in \mathbb{R}$, $\forall t_1 \dots t_k \in \mathbb{Z}$, $\forall k \in \mathbb{N}_+$

<u>Def. 2</u> A SP $X(t)$ is stationary (in weak sense) if

(a) $m_X(t) = m_X(t+\tau)$ $\quad \forall \tau \in \mathbb{Z}$

(b) $R_X(t,s) = R_X(t+\tau, s+\tau)$ $\quad \forall \tau \in \mathbb{Z}$

(a) $\Rightarrow$ $m_X(t)$ is constant : $m_X(t) = m_X$ $\forall t$

(b) $\Rightarrow$ $R_X(t,s) = R_X(t-s)$

$\tau = t-s$ "lag"

$R_X(0)$ is the variance (and it is constant)

For stationary SPs we consider

→ the mean $m_X \in \mathbb{R}$

→ the covariance function

$$R_X(\tau) = E\left[\left(X(t+\tau) - m_X\right)\left(X(t) - m_X\right)\right]$$

<u>Cross-covariance</u>

Given two SPs $X(t)$ and $Y(t)$ we define the cross-covariance function

$$R_{XY}(t,s) = E\left[\left(X(t) - m_X(t)\right)\left(Y(s) - m_Y(s)\right)\right]$$

If X and Y are jointly stationary

$$R_{XY}(\tau) = E\left[\left(X(t+\tau) - m_X\right)\left(Y(t) - m_Y\right)\right]$$

Vector SPs (stationary)

$$X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_m(t) \end{bmatrix}$$

mean $\quad m_X = E\left[X(t)\right] \in \mathbb{R}^m$

covariance function

$$R_X(\tau) = E\left[\left(X(t+\tau) - m_X\right)\left(X(t) - m_X\right)^T\right]$$

cross-covariance function of two SPs $X(t) \in \mathbb{R}^n$ & $Y(t) \in \mathbb{R}^m$

$$R_{XY}(\tau) = E\left[\left(X(t+\tau) - m_X\right)\left(Y(t) - m_Y\right)^T\right]$$

Properties of the covariance function

1) $R_X(\tau) = R_X^T(-\tau)$

2) $R_X(\emptyset)$ has positive elements in its diagonal

3) $$\begin{bmatrix} R_X(0) & R_X(1) & \cdots & R_X(m-1) \\ R_X(-1) & R_X(0) & \cdots & R_X(m-2) \\ \vdots & & & \\ R_X(-m+1) & R_X(-m+2) & \cdots & R_X(0) \end{bmatrix}$$

is symmetric and positive ~~definite~~ semidefinite $\forall m \geq$

4) $R_{XY}(\tau) = R_{YX}^T(-\tau)$

5) For a scalar $X(t)$

$$|R_X(\tau)| \leq R_X(\emptyset) \qquad \forall \tau \in \mathbb{Z}$$

White process

A SP $X(t)$ is said to be WHITE if the variables $X(t_i)$, $t_i \in \mathbb{Z}$ are all independent.

Its covariance function is given by

$$R_X(t,s) = \begin{cases} R_X(t,t) & \text{if } t=s \\ \emptyset & \text{if } t \neq s \end{cases}$$

If $m_X(t) = m_X$ and $R_X(t,t) = \sigma_X^2$ then the white process $X(t)$ is stationary and we denote it by

$$X(t) \sim WN(m_X, \sigma_X^2)$$