Gaussian RVs

$X \in \mathbb{R}^n$

pdf

$$f_X(x) = \frac{1}{\sqrt{(2\pi)^n \cdot \det P}} e^{-\frac{1}{2}(x-m)^T P^{-1}(x-m)}$$

$m \in \mathbb{R}^n$, $P = P^T > 0$, $P \in \mathbb{R}^{n \times n}$

Properties

1. $E[X] = m$
2. $E[(X-m)(X-m)^T] = P$
3. $X \sim f_X(x)$

   In the Gaussian case we write

   $X \sim N(m, P)$

   Then, $Y = AX + b$ satisfies

   $Y \sim N(A \cdot m + b, APA^T)$

4. If X, Y are Gaussian and

$$E\left[(X-m_X)(Y-m_Y)^T\right] = 0$$

then X and Y are independent.

5. Central limit theorem

$X_1, X_2, \ldots X_n$ independent RVs

$$E[X_i] = m_i \qquad\qquad i = 1, \ldots, n$$
$$E\left[(X_i - m_i)^2\right] = \sigma_i^2$$

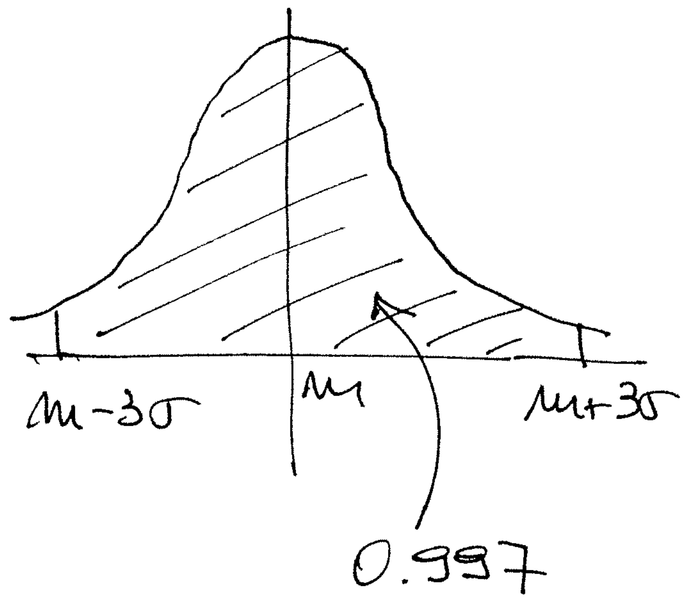
Let $$Z_n = \frac{\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} m_i}{\sqrt{\sum_{i=1}^{n} \sigma_i^2}}$$

$$\lim_{n \to +\infty} f_{Z_n}(z) = \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}}_{N(0,1)}$$

For Gaussian RV, the confidence intervals are defined as

$$\Pr\{m-k\sigma < X \leq m+k\sigma\} = \alpha$$

k	α
1	0.68
2	0.95
3	0.997



Function of RVs

X scalar RV $\qquad g : \mathbb{R} \to \mathbb{R}$

$Y = g(X)$

$$f_Y(y) = \sum_{i=1}^{m} \frac{f_X(x_i)}{|g'(x_i)|}$$

where

$$g(x_1) = g(x_2) = \dots = g(x_m) = y$$

$$g'(x) = \frac{d}{dx} g(x)$$

$X \in \mathbb{R}^n \qquad g : \mathbb{R}^n \to \mathbb{R}^n$

$Y = g(X) \in \mathbb{R}^n$

$$f_Y(y) = \sum_{i=1}^{m} \frac{f_X(x_i)}{|\det J(x_i)|}$$

where

$g(x_1) = g(x_2) = \dots = g(x_m) = y$

and

$J(x) = \dfrac{\partial g}{\partial x} \in \mathbb{R}^{n \times n}$

is the Jacobian matrix of g

Conditional distributions

Two RVs $X, Y \sim f_{X,Y}(x,y)$

We know that, "a priori"

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\, dy$$

Suppose I observe $Y = y$.

Q: How the pdf of X is affected by this observation?

Definition

The "a posteriori" pdf of X, given $Y = y$ is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

Conditional mean

$$m_{X|Y} = \int_{-\infty}^{+\infty} x \cdot f_{X|Y}(x|y)\, dx =$$

$$= E[X|Y]$$

Conditional variance

$$\sigma^2_{X|Y} = \int_{-\infty}^{+\infty} (x - m_{X|Y})^2 \cdot f_{X|Y}(x|y)\, dx$$

Conditional distribution:
Gaussian case.

Let
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \sim N\left(\begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \begin{pmatrix} P_1 & P_{12} \\ P_{12}^T & P_2 \end{pmatrix} \right)$$

Then
$$f_{x_1|x_2} \sim N\left(m_{x_1|x_2}, P_{x_1|x_2} \right)$$

where
$$m_{x_1|x_2} = m_1 + P_{12} P_2^{-1} (x_2 - m_2)$$

$$P_{x_1|x_2} = P_1 - P_{12} P_2^{-1} P_{12}^T$$