Random Variable

$$X : \Omega \to \mathbb{R}$$

Ω : space of elementary events

Probability

$P\{X \leq x\}$: probability that X takes values $\leq x \in \mathbb{R}$

Cumulative Distribution Function (CDF)

$$F_X(x) = P\{X \leq x\} : \mathbb{R} \to [0,1]$$

1) $\lim_{x \to -\infty} F_X(x) = \emptyset$

2) $\lim_{x \to +\infty} F_X(x) = 1$

3) $F_X(x)$ is a non-decreasing function

4) $P\{a < X \leq b\} = F_X(b) - F_X(a)$

Probability density function (pdf)

$$f_X(x) = \frac{\partial}{\partial x} F_X(x)$$

Properties

1) $f_X(x) \geq 0 \quad \forall x \in \mathbb{R}$

2) $\int_{-\infty}^{+\infty} f_X(x)\, dx = 1$

3) $P\{a < X \leq b\} = \int_a^b f_X(x)\, dx$

Famous examples

(a) Gaussian RV

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{(x-m)^2}{\sigma^2}}$$

$m \in \mathbb{R}, \quad \sigma^2 > 0$

(b) Uniform RV

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a < x \leq b \\ 0 & \text{elsewhere} \end{cases}$$

Multivariate distributions

$$X : \Omega \to \mathbb{R}^n \qquad X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}$$

Joint CDF

$$F_X(x) = P\{X_1 \leq x_1, X_2 \leq x_2, \dots X_n \leq x_n\}$$

$$F_X(x) : \mathbb{R}^n \to [0,1]$$

Joint PDF

$$f_X(x) = \frac{\partial^n}{\partial x_1 \partial x_2 \dots \partial x_n} F_X(x)$$

Marginal pdfs

$$f_{X_1}(x_1) = \underbrace{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty}}_{n-1 \text{ times}} f_X(x)\, dx_2\, dx_3 \dots dx_n$$

Mean and Variance

$X \in \mathbb{R}$

The <u>mean</u> of X is given by

$$E[X] = \int_{-\infty}^{+\infty} x \cdot f_X(x)\, dx \triangleq m_X$$

Given $g(x) : \mathbb{R} \to \mathbb{R}$

$$E[g(X)] = \int_{-\infty}^{+\infty} g(x) \cdot f_X(x)\, dx$$

The variance of X is given by

$$E[(X - m_X)^2] = \int_{-\infty}^{+\infty} (x - m_X)^2 \cdot f_X(x)\, dx$$

Confidence interval of X:

$$P\{m_X - k\sigma_X < X < m_X + k\sigma_X\}$$

$k = 1, 2, 3, \ldots$

$X \in \mathbb{R}^n$ (vector of RVs)

Mean

$$E[X] = \underbrace{\int_{-\infty}^{+\infty}}_{n \text{ times}} x \cdot f_X(x) \underbrace{dx}_{dx_1 dx_2 \ldots dx_n}$$

$$= \begin{bmatrix} E[X_1] \\ E[X_2] \\ \vdots \\ E[X_n] \end{bmatrix} = m_X$$

Covariance Matrix of $X \in \mathbb{R}^n$

$$E\left[(X - m_X)(X - m_X)^T\right] =$$

$$= \begin{pmatrix} E\left[(X_1 - m_{X_1})^2\right] & E\left[(X_1 - m_{X_1})(X_2 - m_{X_2})\right] & \cdots \\ E\left[(X_2 - m_{X_2})(X_1 - m_{X_1})\right] & E\left[(X_2 - m_{X_2})^2\right] & \cdots \\ \vdots & & \ddots \\ E\left[(X_n - m_{X_n})(X_1 - m_{X_1})\right] & E\left[(X_n - m_{X_n})(X_2 - m_{X_2})\right] & \cdots \end{pmatrix}$$

$E\left[(X_1 - m_{X_1})(X_2 - m_{X_2})\right]$ is the cross-covariance of X_1 and X_2

Relationships between RVs

Two RVs X_1 and X_2 are independent if

$$f_{X_1 X_2}(x_1, x_2) = f_{X_1}(x_1) \cdot f_{X_2}(x_2)$$

Two RVs X_1 and X_2 are uncorrelated if

$$E\left[(X_1 - m_{X_1})(X_2 - m_{X_2})\right] = \emptyset$$

Independency $\Rightarrow$ Uncorrelation
(the vice versa is true only for Gaussian variables)