System Identification and Data Analysis
Lab session # 7

The evolution of a cattle breeding can be described by a population dynamic model in input-state-output representation, in which a state variable represents the number of individuals in the corresponding class.

We consider a model with the following classes:

x_1	:	young females	x_4	:	young males
x_2	:	adult females	x_5	:	adult males
x_3	:	old females	x_6	:	old males

If $x_i(k)$ represents the number of heads in the i-th class during the year k, the population evolution is described by the equations

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \\ x_5(k+1) \\ x_6(k+1) \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 & 0 & 0 & 0 \\ b_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_2 & b_3 & 0 & 0 & 0 \\ c_1 & c_2 & c_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & d_2 & d_3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \\ x_5(k) \\ x_6(k) \end{bmatrix} + \begin{bmatrix} u_1(k) \\ u_2(k) \\ u_3(k) \\ u_4(k) \\ u_5(k) \\ u_6(k) \end{bmatrix} + \begin{bmatrix} w_1(k) \\ w_2(k) \\ w_3(k) \\ w_4(k) \\ w_5(k) \\ w_6(k) \end{bmatrix}$$

where

- the coefficients a_i and c_i, $i = 1, 2, 3$, are the birthrates of female and male heads, respectively, from cows belonging to the class x_i;

- the coefficients b_i and d_i, $i = 1, 2, 3$, are the surviving rates of female heads of the class x_i and of male heads of the class x_{i+3}, respectively;

- $u_i(k)$ is the buying and selling balance of heads in the i-th class, during the year k;

- $w_i(k)$ represents the discrepancy between the true evolution of the i-th class and the nominal one predicted by the model.

Assume that we have data from an annual census of the cattle breeding, reporting the total number of female and male heads over a period of 101 years

$$\begin{aligned} y_1(k) &= x_1(k) + x_2(k) + x_3(k) + v_1(k) \\ y_2(k) &= x_4(k) + x_5(k) + x_6(k) + v_2(k) \end{aligned} \qquad k = 0, 1, \ldots, 100$$

where the variables $v_j(k)$, $j = 1, 2$, represent the errors made in the census.

A. Assume that $w_i(k)$, $i = 1, \ldots, 6$ and $v_j(k)$, $j = 1, 2$, can be modeled as realizations of discrete-time stochastic white processes, independent with each other, with zero mean and such that

- the variance of processes $w_i(k)$ is equal to σ_w^2;

- the variance of processes $v_j(k)$ is equal to σ_v^2.

Design and implement a Kalman filter providing an estimate of the population in each class, by using the input data $u(k)$ and the measurements $y(k)$. Plot the obtained estimates and compare them with the true evolution of each class. Compute the state estimation error $\sqrt{\|x(k) - \hat{x}(k|k)\|^2}$ and compare it with the square root of the trace of the covariance matrix $P(k|k)$. For each state variable, compare the estimation error $x_i(k) - \hat{x}_i(k|k)$ with the corresponding confidence interval $\pm 3\sqrt{P_{ii}(k|k)}$.

B. By using the Matlab function `dlqe`, design and implement the asymptotic Kalman filter relative to the same problem of exercise A. Compare the estimates provided by the asymptotic Kalman filter to those of the time-varying Kalman filter used in exercise A. Discuss the role of the Kalman filter initialization.

C. Consider the same evolution of the state variables as in exercise A (with the same input signal $u(k)$, but assume that a different set of measurements $y(k)$ is available. The measurement errors affecting these new data have a standard deviation equal to

$$\sigma_{v_j}(k) = 300 \cdot 0.95^k \qquad j = 1, 2.$$

Apply the Kalman filter again, using these new measurements. Compare the estimates with those provided by a Kalman filter assuming a constant standard deviation for the measurement noise, equal to 300. Evaluate the estimation errors at each time instant in both cases, and compare them to the corresponding confidence intervals provided by the Kalman filter.

D. Assume now that a new input-output data set is available, relative to the same model as in exercise A, but with a time-varying parameter a_2 such that

$$a_2(k) = 0.45 + 0.3 \sin\left(\frac{2\pi}{20}k\right).$$

Apply the Kalman filter and compare the results to those obtained by using the filter adopted in exercise A (in which matrix A is constant). Evaluate the true state estimation errors of both filters and compare them to the corresponding confidence intervals provided by the filters.

Data.

Data contained in the file `data_labsession7.mat` refer to the following variables:

- parameters of the model: `a1,a2,a3,b1,b2,b3,c1,c2,c3,d1,d2,d3`;
- process variances σ_w^2 and σ_v^2: `sw2` and `sv2`, respectively;
- input `uA` and output `yA` sequences to be used in exercises A and B: the i-th column refers to the i-th variable (e.g., `uA(:,1)` corresponds to $u_1(t)$ for $t = 0, 1, \ldots, 100$);
- sequences of true state variables `xtrueA` in exercises A, B and C, to be used *uniquely* to compute the true state estimation errors;
- input `uA` and output `yC` sequences to be used in exercise C;
- input `uD`, output `yD` and true state `xtrueD`, to be used in exercise D;
- vector of times `t` (referred to all involved signals).