Chapter 3

Sliding mode control

In general, two main sources of uncertainty may affect a model: structured and unstructured uncertainty. While structured uncertainty refers to inaccuracies in some components of the system, unstructured one usually regards unmodeled dynamics. Both these uncertainties play a role in controlling a system; in fact, their presence may cause strong undesirable effects, especially when dealing with nonlinear systems.

In this chapter, a technique for robust controller design for nonlinear systems is reported. We refer to this technique to as *sliding mode control* (SMC).

3.1 Illustrative example

The main idea of the SMC technique is reported by referring to the following example. Let us consider the nonlinear system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = f(\mathbf{x}) + u \end{cases} \tag{3.1}$$

where $\mathbf{x} = [x_1, x_2]'$ denotes the state vector, u is the control input, and $f(\mathbf{x})$ is a generic nonlinear function. Although function f may depend also on time and on other exogenous variables, for ease of notation we explicitly report only the dependence on the state vector.

Let the initial conditions be

$$\begin{cases} x_1(0) = x_{10} \\ x_2(0) = x_{20} \end{cases}$$

Assuming an exact knowledge of the states at each time, the goal is to design a control input u which drives the states to the origin.

If the nonlinear function $f(\mathbf{x})$ is known, one may apply the feedback linearization techniques described in the previous chapter. Setting as output $y = x_1$, and applying the input-output feedback linearization, one has

$$\begin{cases} \dot{y} = \dot{x}_1 = x_2 \\ \ddot{y} = \dot{x}_2 = f(\mathbf{x}) + u \end{cases}$$

Since the relative degree is equal to the order of the system, there is no internal dynamics. The control input can be designed as $u = v - f(\mathbf{x})$ obtaining $\ddot{y} = v$. Assuming that the desired output is $y_d = 0$, the equivalent input v can be chosen as

$$v = -k_1\dot{y} - k_0 y$$

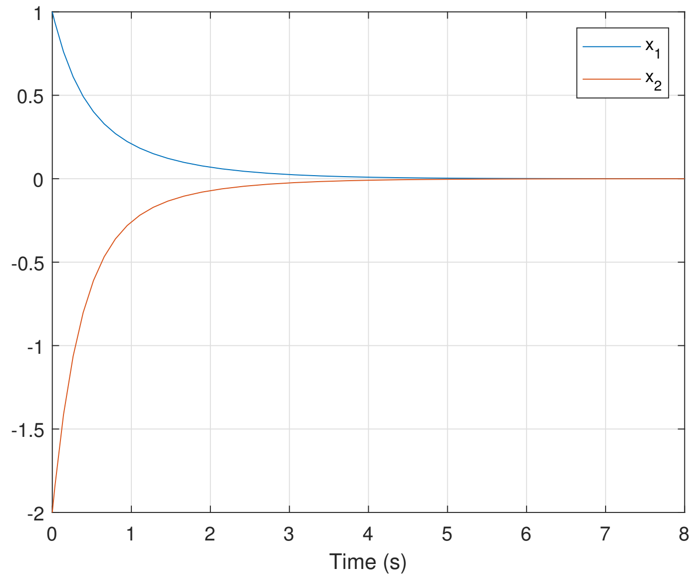

Figure 3.1: Time plot of the states for the controller based on feedback linearization (exact knowledge of $f(\mathbf{x})$).

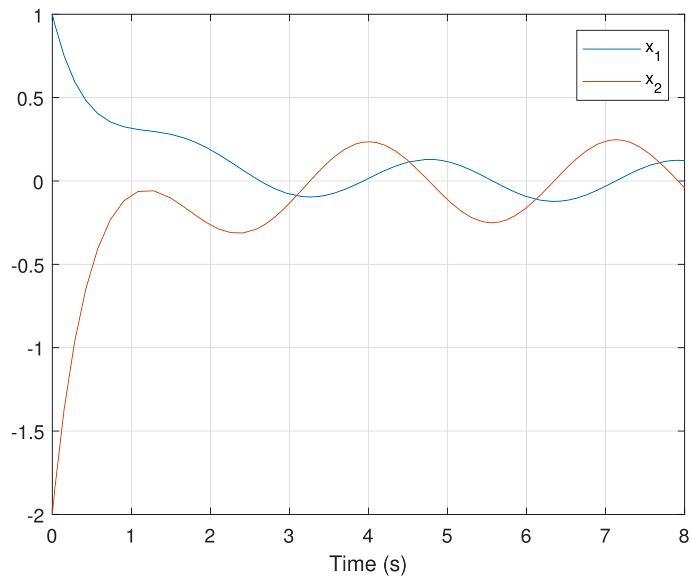

Figure 3.2: Trajectories of the states for the controller based on feedback linearization (ignoring the disturbance $f(\mathbf{x})$).

obtaining

$$\ddot{y} + k_1\dot{y} + k_0 y = 0$$

The controller parameters must be chosen to asymptotically stabilize the system. A possible choice is $k_1 = 3,\ k_2 = 4$. In Fig. 3.1, the time plot of the states are reported assuming $\mathbf{x}(0) = [1, -2]'$. As expected, the states converge asymptotically to zero.

Let us now assume that $f(\mathbf{x})$ represents an unknown disturbance. Since $f(\mathbf{x})$ is unknown, a possible choice is to ignore it, obtaining a controller $u = v = -k_1\dot{y} - k_0 y$. If the actual disturbance is

$$f = \cos(2t)$$

the states will not converge to zero, but they remain limited, as illustrated in Fig. 3.2.

Now, consider the case that we know a bound on the unknown function $f(\mathbf{x})$, that is

$$|f(\mathbf{x})| \leq F \quad , \quad F > 0$$

In this example, one may set $F = 1$. The sliding mode control aims at controlling the system assuming the knowledge of a bound on f.

Suppose the input u has been designed to enforce the following first-order dynamics

$$\dot{x}_1 + \lambda x_1 = 0 \quad , \quad \lambda > 0 \tag{3.2}$$

Since $\dot{x}_1 = x_2$, the solution of (3.2) is

$$\begin{cases} x_1(t) = x_{10}\, e^{-\lambda t} \\ x_2(t) = -\lambda\, x_{10}\, e^{-\lambda t} \end{cases} \tag{3.3}$$

where both the states converge asymptotically to 0. Notice that this result is independent on f. The key point is how to design u such that (3.2) holds.

Let us define the *sliding variable* σ as

$$\sigma = \sigma(\mathbf{x}) = x_2 + \lambda x_1 \quad , \quad \lambda > 0 \tag{3.4}$$

To allow $\mathbf{x}$ to asymptotically converge to $\mathbf{0}$ with the convergence rate as in (3.3), one must enforce that condition $\sigma = 0$ must be achieved <u>in finite time</u>. To prove this fact, we may take advantage of Lyapunov function techniques. Let us compute the derivative of σ

$$\begin{cases} \dot{\sigma} = \dot{x}_2 + \lambda \dot{x}_1 = \lambda x_2 + f(\mathbf{x}) + u \\ \sigma(0) = \sigma_0 \end{cases} \tag{3.5}$$

Let us choose the Lyapunov candidate function as

$$V(\sigma) = \frac{\sigma^2}{2} \tag{3.6}$$

Notice that, the Lyapunov function V denotes the (half) square distance of σ from the surface $\sigma = 0$. It is known that a sufficient condition to guarantee asymptotic stability of (3.5) is

$$\dot{V}(\sigma) < 0 \quad , \quad \forall \sigma \neq 0 \tag{3.7}$$

Condition (3.7) guarantees that $\sigma \to 0$ for $t \to \infty$. However, we are interested in achieving $\sigma = 0$ in <u>finite</u> time. To this aim, the following proposition is introduced.

Proposition 3.1. *Let*

$$\dot{V} \leq -\alpha \sqrt{V} \quad , \quad \alpha > 0 \tag{3.8}$$

hold. Then, (3.5) *goes to zero in finite time.*

Proof. It is easy to note that (3.8) implies (3.7), so asymptotic convergence is assured. Let $V_0 = V(0)$ and $V_t = V(t)$. By separating variables and integrating (3.8) in the time interval $[0, t]$, one has

$$\int_{V_0}^{V_t} \frac{1}{\sqrt{V}}\, dV \leq \int_0^t -\alpha\, d\tau$$

which leads to

$$\sqrt{V(t)} \leq -\frac{\alpha}{2} t + \sqrt{V_0}$$

So, V reaches zero in a time which is no greater than

$$t_{reach} = \frac{2\sqrt{V_0}}{\alpha} \tag{3.9}$$

□

If the control u is such that (3.8) holds, then the sliding variable σ will reach zero in finite time, and it will remain there forever.

In this example, by (3.5) the derivative of V turns out to be

$$\dot{V} = \sigma\dot{\sigma} = \sigma(\lambda x_2 + f(\mathbf{x}) + u)$$

Let us choose $u = -\lambda x_2 + v$, one has

$$\dot{V} = \sigma f(\mathbf{x}) + \sigma v \leq |\sigma| F + \sigma v \tag{3.10}$$

Let sign$(\cdot)$ denote the "sign" function, that is

$$\text{sign}(a) = \begin{cases} +1, & \text{if } a > 0 \\ 0, & \text{if } a = 0 \text{ (actually, we are not interested in this case)} \\ -1, & \text{if } a < 0 \end{cases}$$

By choosing

$$v = -\rho\,\text{sign}(\sigma) \quad , \quad \rho > 0 \tag{3.11}$$

(3.10) becomes

$$\dot{V} \leq |\sigma| F - \rho|\sigma| = -|\sigma|(\rho - F)$$

Notice that, by substituting (3.6) in (3.8) one has

$$\dot{V} \leq -\alpha\sqrt{V} = -\frac{\alpha}{\sqrt{2}}|\sigma| \quad , \quad \alpha > 0 \tag{3.12}$$

To satisfy (3.8), one must impose

$$-|\sigma|(\rho - F) \leq -\frac{\alpha}{\sqrt{2}}|\sigma|$$

and so the gain ρ results

$$\rho \geq F + \frac{\alpha}{\sqrt{2}} \tag{3.13}$$

Thus, the command u which drives the sliding variable to zero in finite time is

$$u = -\lambda x_2 - \rho\,\text{sign}(\sigma) \tag{3.14}$$

Remark. Notice that, the above reasoning is based on the fact that $\dot{\sigma}$ is a function of u. This aspect must be taken into account when designing the sliding variable σ.

Remark. The first component of (3.13) allows one to compensate the disturbance $f(\mathbf{x})$, while the second one is related to the time needed to σ to reach zero. According to (3.9), the greater is α, the shorter is the time.

The equation

$$\sigma = x_2 + \lambda x_1 = 0 \quad , \quad \lambda > 0$$

is called *sliding surface.*

By (3.12), one has that condition (3.8) is equivalent to

$$\dot{V} = \sigma\dot{\sigma} \leq -\frac{\alpha}{\sqrt{2}}|\sigma| \tag{3.15}$$

which is called *reachability condition.* If this condition is satisfied, then the system trajectory is driven towards the sliding surface, and it will remain on it thereafter.

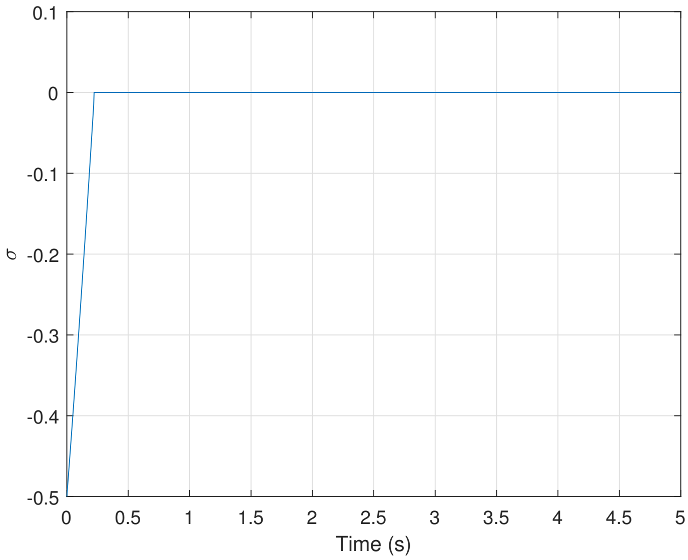

Figure 3.3: Time plot of the sliding variable σ assuming $\lambda = 1.5$.

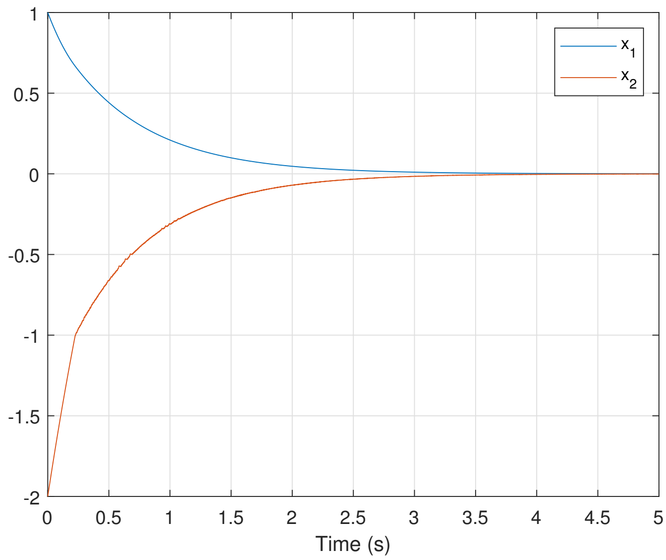

Figure 3.4: Trajectories of the states for sliding mode control.

Let us simulate the system in (3.1) with a sliding mode controller with $\rho = 2$ and $\lambda = 1.5$. Let $\mathbf{x}(0) = [1, -2]'$ and let $f(\mathbf{x}, t) = \sin(2t)$. In Fig. 3.3, one may observe the behavior of the sliding variable. After a finite time, the system states reach the sliding surface, after that they remain on it. In Fig. 3.4, the trajectory of the two states are depicted. Notice the discontinuity around $t = 0.22$ when the sliding surface is reached.

The state trajectories on the phase diagram are depicted in Fig. 3.5. Two steps are shown: the former (*reaching phase*) is related to $\sigma \neq 0$, where condition $\sigma\dot{\sigma} \leq -\frac{\alpha}{\sqrt{2}}|\sigma|$ guarantees finite-time reaching of the sliding surface. Once the sliding surface is reached, the *sliding phase* allows exponential convergence to $\mathbf{x} = \mathbf{0}$.

A main issue affecting SMC in real implementations is the so-called *chattering* phenomena, that is high-frequency oscillations (zigzag) of the command input and of the system states. This fact, as illustrated in Fig. 3.6, is due to the imprecise measurements or imperfect implementation of the sign function in (3.14). In Section 3.3, some solutions aimed at reducing the chattering effect will be described.

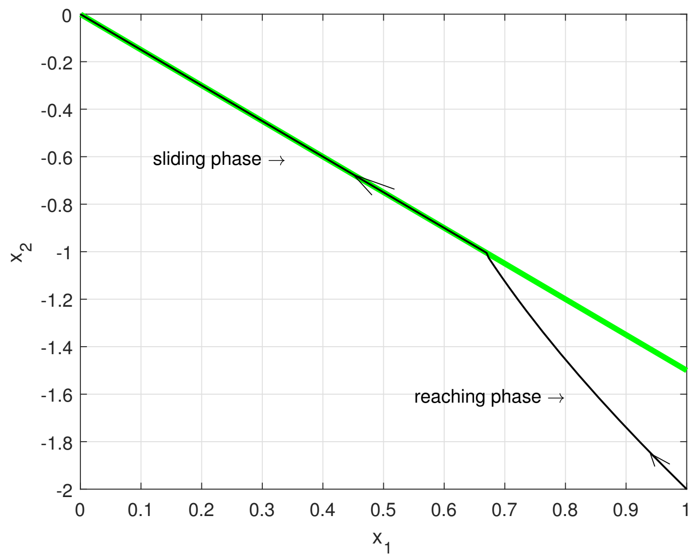

Figure 3.5: Phase diagram showing the trajectory of the sliding variable. The green line denotes the sliding surface.

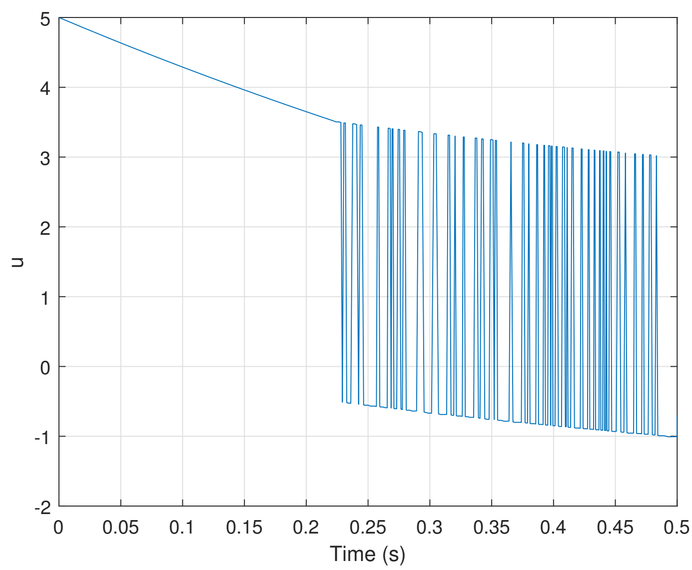

Figure 3.6: Chattering phenomena on the command input.

3.2 General procedure and tracking

In Section 3.1, an example involving a second-order system has been reported. Let us now consider the general case of a nonlinear system of order n in companion form,

$$x^{(n)} = f(t, \mathbf{x}) + g(t, \mathbf{x})u$$

where $\mathbf{x} = [x, \dot{x}, \ldots, x^{(n-1)}]' = [x_1, x_2, \ldots, x_n]'$ denotes the state vector. Notice that, functions f and g can depend both on the state and on time. Hereafter, to simplify the notation, the dependence on t (and sometimes on $\mathbf{x}$) will be omitted.

With a slight abuse of notation, the sliding variable σ can be designed as follows

$$\sigma = \left(\frac{d}{dt} + \lambda\right)^{n-1} x \quad , \quad \lambda > 0 \tag{3.16}$$

For $n = 2$ and 3, notation (3.16) means

$$\sigma = \dot{x} + \lambda x = x_2 + \lambda x_1 \quad , \quad n = 2$$

$$\sigma = \ddot{x} + 2\lambda\dot{x} + \lambda^2 x = x_3 + 2\lambda x_2 + \lambda^2 x_1 \quad , \quad n = 3$$

and so on.

Since σ depends on the last state variable $x_n = x^{(n-1)}$, it will be sufficient to derive once in order to obtain an explicit dependence on the command u. For instance, if $n = 3$ one has

$$\dot{\sigma} = x^{(3)} + 2\lambda\ddot{x} + \lambda^2\dot{x} = \dot{x}_3 + 2\lambda x_3 + \lambda^2 x_2 = f(\mathbf{x}) + g(\mathbf{x})u + \underbrace{2\lambda x_3 + \lambda^2 x_2}_{r(\mathbf{x})}$$

Then, for a system of order n the expression of $\dot{\sigma}$ is

$$\dot{\sigma} = f(\mathbf{x}) + g(\mathbf{x})u + r(\mathbf{x}) \tag{3.17}$$

where $r(\mathbf{x})$ changes with n according to (3.16). Notice that, assuming to exactly know the state vector $\mathbf{x}$, then $r(\mathbf{x})$ is known, too.

3.2.1 Uncertainty on $f(\mathbf{x})$

Assume $f(\mathbf{x})$ is uncertain, that is

$$f = \widehat{f} + \widetilde{f}$$

where $\widehat{f}$ denotes the nominal value and $\widetilde{f}$ the corresponding uncertainty, for which a given bound is known

$$|\widetilde{f}| = |f - \widehat{f}| \leq F$$

Assuming $g(\mathbf{x}) \neq 0$ in the considered domain, according to the reasoning of Section 3.1, it is possible to choose u as

$$u = \frac{1}{g(\mathbf{x})}(-\widehat{f}(\mathbf{x}) - r(\mathbf{x}) - \rho\,\mathrm{sign}(\sigma)) \tag{3.18}$$

for a suitable large ρ.

In case we are interested in the tracking problem, let us denote the reference vector as $\mathbf{x}_d = [x_d, \dot{x}_d, \ldots, x_d^{(n-1)}]' = [x_{1d}, x_{2d}, \ldots, x_{nd}]'$. The aim is to drive the states towards the desired ones, i.e., $\mathbf{x} \to \mathbf{x}_d$. In this case, the sliding variable can be chosen as in (3.16) by replacing the state vector with the error vector $\mathbf{e} = \mathbf{x} - \mathbf{x}_d$, that is

$$\sigma = \left(\frac{d}{dt} + \lambda\right)^{n-1} e \quad , \quad \lambda > 0 \tag{3.19}$$

In this case, it is assumed that the reference x_d and all its derivatives up to time n are known at each time. So, the sliding dynamics (3.17) becomes

$$\dot{\sigma} = f(\mathbf{x}) + g(\mathbf{x})u - \dot{x}_{nd} + r(\mathbf{e}) \tag{3.20}$$

where $r(\mathbf{e})$ depends on $e_2, \ldots, e_n$. Similar to (3.18), the control signal becomes

$$u = \frac{1}{g(\mathbf{x})}\left(\widehat{u} - \rho\,\mathrm{sign}(\sigma)\right) \tag{3.21}$$

where

$$\widehat{u} = -\widehat{f}(\mathbf{x}) + \dot{x}_{nd} - r(\mathbf{e}) \tag{3.22}$$

Substituting (3.21) into (3.20), one has

$$\dot{\sigma} = \widetilde{f} - \rho\,\mathrm{sign}(\sigma)$$

By choosing the Lyapunov function as in (3.6), i.e., $V = \sigma^2/2$ one has

$$\dot{V} = \sigma\dot{\sigma} = (\widetilde{f} - \rho\,\mathrm{sign}(\sigma))\sigma = \widetilde{f}\sigma - \rho|\sigma| \le |\sigma|(F - \rho)$$

So, it is sufficient to choose

$$\rho = F + \eta \quad , \quad \eta > 0 \tag{3.23}$$

to guarantee asymptotic stability. So, the overall command input u results

$$u = \frac{1}{g(\mathbf{x})}\Big(-\widehat{f}(\mathbf{x}) + \dot{x}_{nd} - r(\mathbf{e}) - (F + \eta)\,\mathrm{sign}(\sigma)\Big) \quad , \quad \eta > 0 \tag{3.24}$$

The following proposition states that bounds on σ can be interpreted as bounds on the tracking error $\mathbf{e}$, and so the sliding variable σ represents a measure of the tracking performance.

Proposition 3.2. *Assume* $\mathbf{e}(0) = 0$ *and let* $|\sigma(t)| \le \delta$, $\forall t \ge 0$. *Then,*

$$|e^{(i)}(t)| \le \frac{(2\lambda)^i}{\lambda^{n-1}}\delta \quad , \quad i = 0, \ldots, n-1, \quad t \ge 0. \tag{3.25}$$

In case $\mathbf{e}(0) \ne 0$, bound (3.25) holds asymptotically, after a transient.

Example 3.1. Let us consider the following second-order system

$$\ddot{x} + a(t)\dot{x}^2\cos(3x) = u \tag{3.26}$$

where $a(t)$ is unknown but it is bounded in $1 \le a(t) \le 2$. Let us define

$$f(\mathbf{x}) = -a(t)\dot{x}^2\cos(3x)$$

Thus, (3.26) becomes

$$\ddot{x} = f(\mathbf{x}) + u$$

To minimize the uncertainty on f, since $1 \leq a(t) \leq 2$, one may define

$$\widehat{f}(\mathbf{x}) = -1.5\,\dot{x}^2 \cos(3x)$$

Let $\widetilde{f}(\mathbf{x}) = f(\mathbf{x}) - \widehat{f}(\mathbf{x})$, then

$$|\widetilde{f}(\mathbf{x})| \leq 0.5\,\dot{x}^2 |\cos(3x)| \triangleq F(\mathbf{x})$$

So, the system can be rewritten as

$$\ddot{x} = \widetilde{f}(\mathbf{x}) + \widehat{f}(\mathbf{x}) + u$$

Assume we are interested in tracking the first state variable, i.e. $x \simeq x_d$. By defining the error signal $e = x - x_d$, one may design the sliding variable according to (3.19),

$$\sigma = \dot{e} + \lambda e$$

and

$$\dot{\sigma} = \ddot{e} + \lambda\dot{e} = \widetilde{f}(\mathbf{x}) + \widehat{f}(\mathbf{x}) + u - \ddot{x}_d + \lambda\dot{e}$$

By (3.24), the control command can be chosen as

$$u = -\widehat{f}(\mathbf{x}) + \ddot{x}_d - \lambda\dot{e} - (F(\mathbf{x}) + \eta)\,\mathrm{sign}(\sigma) \quad , \quad \eta > 0$$

in order to drive the error to zero asymptotically. △

3.2.2 Uncertainty on $g(\mathbf{x})$

Consider again the nonlinear system

$$x^{(n)} = f(t, \mathbf{x}) + g(t, \mathbf{x})u$$

Now, let us assume that also the term g is affect by uncertainty

$$0 < \underline{g} \leq g \leq \overline{g}$$

Since the control command enters multiplicatively, it is convenient to choose the estimate of g as the geometric mean of its bounds, that is

$$\widehat{g} = \sqrt{\overline{g}\,\underline{g}}$$

By defining

$$\beta = \sqrt{\overline{g}/\underline{g}}$$

one has

$$\frac{1}{\beta} \leq \frac{\widehat{g}}{g} \leq \beta$$

It can be shown that the command which guarantees the asymptotic convergence of the error to 0 is

$$u = \frac{1}{\widehat{g}}\left(\widehat{u} - \rho\,\mathrm{sign}(\sigma)\right) \tag{3.27}$$

with $\widehat{u}$ defined as in (3.22), and

$$\rho \geq \beta(F + \eta) + (\beta - 1)|\widehat{u}| \quad , \quad \eta > 0 \tag{3.28}$$

Notice that, if no uncertainty affects the term g, then $\beta = 1$ and (3.27) coincides with (3.21).

Summarizing, the design of an SMC controller can be divided in two parts:

1. design the first-order sliding surface σ;
2. design the control u which drives the sliding variable σ to 0 in finite time. Notice that to accomplish this task, one needs to compute the dynamics of σ, that is, $\dot{\sigma}$.

Remark. It is worthwhile to note that the system dynamics in the sliding mode do not depend on the functions f and g, but a bound on them is needed to compute ρ in (3.28), and hence to compute the control command u in (3.27). Moreover, it is assumed that the system states are available and the reference signal and its derivatives are known.

3.2.3 Equivalent control

Let us refer to the system in (3.1), and hence to the sliding variable defined in (3.4). Assume that at time t_r the sliding surface $\sigma = x_2 + \lambda x_1 = 0$ is reached, and x_1, x_2 remain on that surface for $t \geq t_r$. Then, $\sigma = \dot{\sigma} = 0$, for all $t \geq t_r$. So,

$$\dot{\sigma} = \lambda x_2 + f(\mathbf{x}) + u = 0$$

This expression requires that the *equivalent input* to be applied (from time t_r onwards) to enforce $\dot{\sigma} = 0$ corresponds to

$$u_{eq} = -\lambda x_2 - f(\mathbf{x}) \tag{3.29}$$

Notice that the equivalent control (3.29) cannot be implemented in a real system, since $f(\mathbf{x})$ is not exactly known. Moreover, the equivalent control u_{eq} is not the actual control which is applied to the system. In fact, the actual control corresponds to

$$u = -\lambda x_2 - \rho\,\mathrm{sign}(\sigma)$$

which is a discontinuous control function. So, the equivalent control can be viewed as the average effect of the high-frequency switching control (3.14). Let us write the actual applied control signal as

$$u = -\lambda x_2 - \rho z_s$$

where $z_s = \mathrm{sign}(\sigma)$. The equivalent input can be estimated online by filtering z_s through a low-pass filter. For instance, by using a first-order low-pass filter, one has

$$G_{LP}(s) = \frac{Z_f(s)}{Z_s(s)} = \frac{1}{1 + \tau s}$$

where z_f is the output of the filter, and τ denotes the time constant of the filter. Denoting with $\widehat{u}_{eq}$ the estimate of the equivalent input, one has

$$\begin{aligned} \dot{z}_f &= -\frac{1}{\tau} z_f + \frac{1}{\tau} z_s \\ \widehat{u}_{eq} &= -\lambda x_2 - \rho z_f \end{aligned} \tag{3.30}$$

We can obtain a good approximation of u_{eq} by choosing τ small. In practical applications, this filter will be implemented in discrete time, and τ would be chosen small but greater than the sampling time.

By comparing (3.29) and (3.30), an estimate of f can be obtained as

$$f_{est} = \rho z_f$$

The equivalent control is useful also to describe the system equations once the sliding surface $\sigma = x_2 + \lambda x_1 = 0$ is reached, i.e., for $t \geq t_r$. Let us refer to the system (3.1) and let us substitute the input u computed in (3.14). One has

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u + f(\mathbf{x}) = -\lambda x_2 - \rho\,\text{sign}(x_2 + \lambda x_1) + f(\mathbf{x}) \end{cases}$$

with

$$\begin{cases} x_1(t_r) = x_{1r} \\ x_2(t_r) = -\lambda x_{1r} \end{cases}$$

From (3.29), by substituting u with u_{eq}, one gets

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u_{eq} + f(\mathbf{x}) = -\lambda x_2 \end{cases}$$

Since $\sigma = x_2 + \lambda x_1 = 0$, one has $x_2 = -\lambda x_1$, and hence

$$\begin{cases} \dot{x}_1 = -\lambda x_1 \\ x_2 = -\lambda x_1 \end{cases}$$

with $x_1(t_r) = x_{1r}$. Thus, once the sliding surface is reached, the system states will evolve as

$$\begin{cases} x_1(t) = x_{1r}\, e^{-\lambda(t-t_r)} \\ x_2(t) = -\lambda x_{1r}\, e^{-\lambda(t-t_r)} \end{cases} \tag{3.31}$$

From (3.31) it is apparent that the rate of convergence of the states to zero is governed by the parameter λ.

3.3 Chattering elimination and attenuation

It has been seen that, in order to apply SMC, the input command u must switch, possibly at high frequencies. Such a command can be safely implemented in some cases, for instance when it is related to the electric voltage driving a motor. In fact, a typical motor control is provided through *pulse width modulation* (PWM), where a high frequency voltage with different duty cycle is involved.

Nevertheless, in general, the chattering behavior depicted in Fig. 3.6 is not desirable from a practical point of view. In fact, it requires a high control activity and it may excite high-frequency unmodeled dynamics. To this purpose, some solutions have been devised to cancel or attenuate the chattering behavior.

3.3.1 Chattering elimination

A possible solution to eliminate chattering consists in approximating the sign function with a smooth one, like, e.g. the *sigmoid* function defined as

$$\text{sigmoid}(x) = \frac{x}{|x| + \varepsilon} \tag{3.32}$$

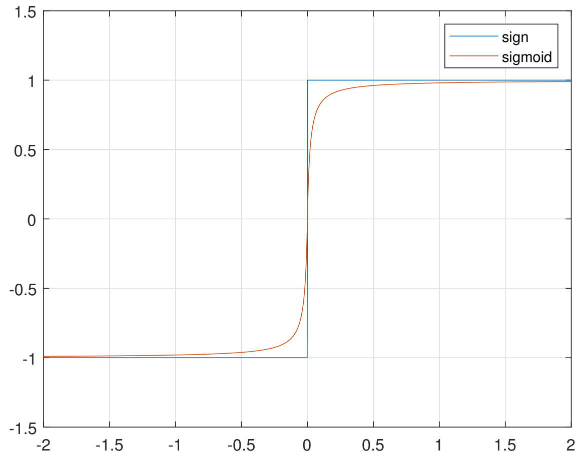

Figure 3.7: Comparison between the sign and the sigmoid functions ($\varepsilon = 0.02$).

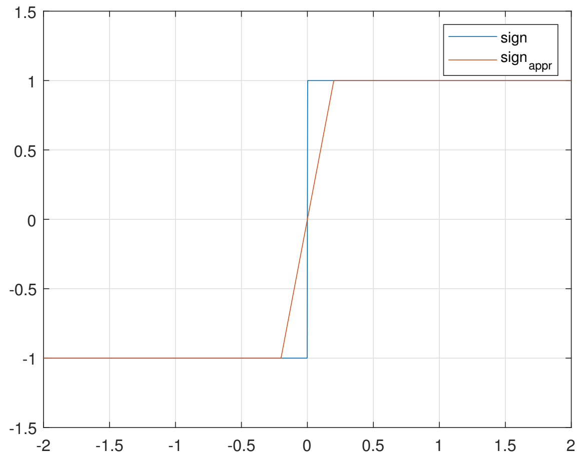

Figure 3.8: Comparison between the sign_{appr} and the sign functions ($B = 0.2$).

with $\varepsilon > 0$ arbitrarily small (see Fig. 3.7). It is clear that the approximation given by (3.32) improves for $\varepsilon \to 0$. So, ε can be designed to achieve the desired trade-off between chattering and performance.

An alternative approximation of the sign function is as follows

$$\text{sign}_{appr}(x) = \begin{cases} \text{sign}(x), & \text{if } |x| \geq B \\ x/B, & \text{if } |x| \leq B \end{cases} \tag{3.33}$$

A comparison between sign_{appr} and the sign function is reported in Fig. 3.8.

Let us consider again the system (3.1) and replace the sign function with the sigmoid (3.32). According to (3.14), the control command u becomes

$$u = -\lambda x_2 - \rho \frac{\sigma}{|\sigma| + \varepsilon}$$

Due to this approximation, the sliding variable cannot reach zero, but it converges around it. The time plot of the command input u for $\varepsilon = 0.02$ is reported in Fig. 3.9. In Fig. 3.10, one may notice that the sliding variable does not converge to zero, but it oscillates around it. The SMC with chattering elimination is usually referred to as *quasi-SMC*, since the sliding variable does not converge to zero in finite time. However, as reported in Fig. 3.11, the system states tends to a neighbor of the origin, whose amplitude decreases as $\varepsilon \to 0$.

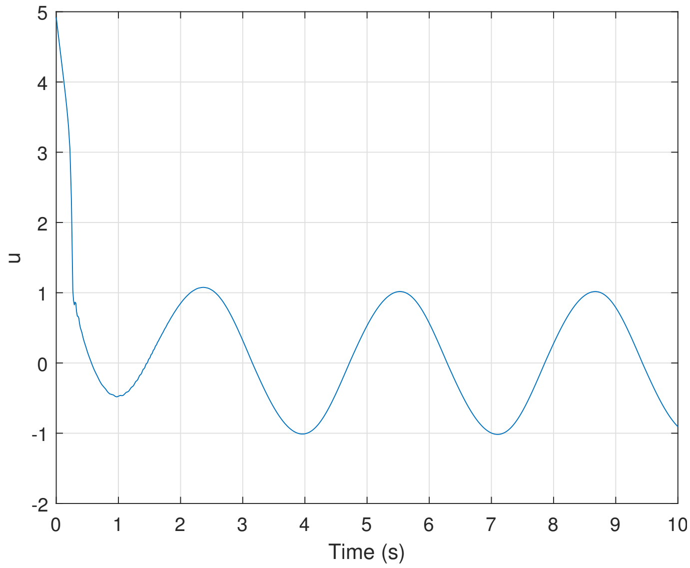

Figure 3.9: Command input with chattering elimination.

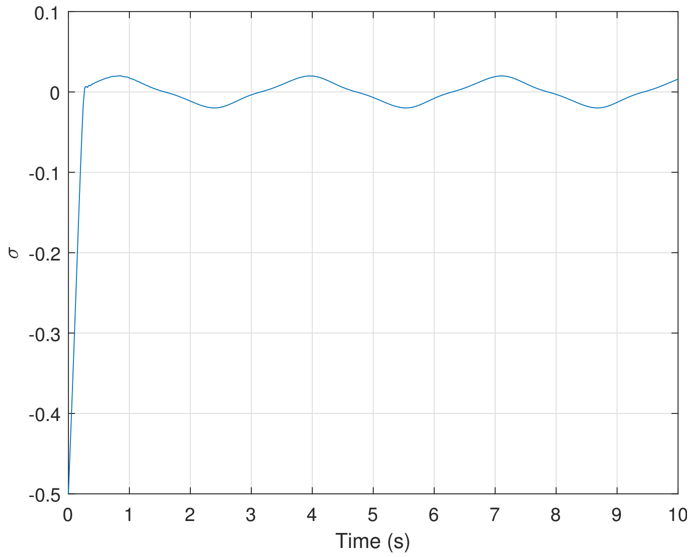

Figure 3.10: Time plot of the sliding variable σ (chattering elimination).

3.3.2 Chattering attenuation

Another method to deal with chattering is to apply SMC in terms of the control function derivative. The actual control will be the integral of the chattering input, and so it will not present discontinuities. This technique is called *chattering attenuation*, since some periodic signals remain after command integration.

Let us rewrite system (3.1) as

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u + f(\mathbf{x}) \\ \dot{u} = \nu \end{cases}$$

with $u(0) = 0$. In addition to $|f(\mathbf{x})| \leq F$, let us assume to know a bound on the derivative of f, that is $|\dot{f}(\mathbf{x})| \leq \overline{F}$.

The sliding variable σ is defined as in (3.4). Similarly, let us introduce the new sliding variable s, defined as

$$s = \dot{\sigma} + \bar{\lambda}\sigma \quad , \quad \bar{\lambda} > 0$$

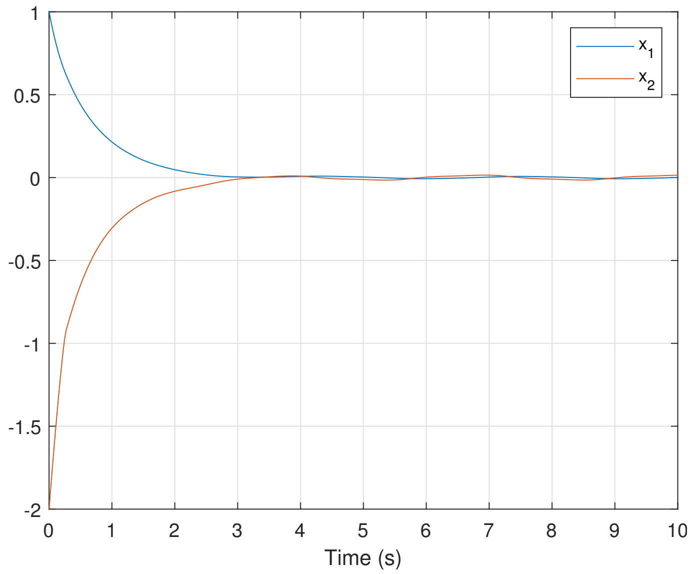

Figure 3.11: Trajectories of the states for sliding mode control with chattering elimination.

The idea is to design a control law ν which provides finite-time convergence of s to 0. This guarantees that $\sigma, \dot{\sigma} \to 0$ exponentially, and hence also $x_1, x_2 \to 0$. Differently from the original formulation of SMC, here $\sigma \to 0$ asymptotically and not in finite-time. So, we name this technique as *asymptotic SMC*.

To design ν, similarly to (3.15), one must impose

$$s\dot{s} \leq -\frac{\alpha}{\sqrt{2}}|s| \tag{3.34}$$

Since $\sigma = x_2 + \lambda x_1$, one has

$$s = \dot{x}_2 + \lambda \dot{x}_1 + \bar{\lambda} x_2 + \bar{\lambda}\lambda x_1 = u + f(\mathbf{x}) + (\lambda + \bar{\lambda})x_2 + \bar{\lambda}\lambda x_1$$

and hence,

$$\dot{s} = \nu + \dot{f}(\mathbf{x}) + (\lambda + \bar{\lambda})(u + f(\mathbf{x})) + \bar{\lambda}\lambda x_2$$

By choosing $\nu = -\bar{\lambda}\lambda x_2 - (\lambda + \bar{\lambda})u + \xi$, one has $\dot{s} = \dot{f}(\mathbf{x}) + (\lambda + \bar{\lambda})f(\mathbf{x}) + \xi$ and hence

$$s\dot{s} = s(\dot{f}(\mathbf{x}) + (\lambda + \bar{\lambda})f(\mathbf{x}) + \xi) \leq |s|(\overline{F} + (\lambda + \bar{\lambda})F) + s\xi$$

Taking $\xi = -\rho\,\mathrm{sign}(s)$ one has

$$s\dot{s} \leq |s|(\overline{F} + (\lambda + \bar{\lambda})F - \rho)$$

To satisfy (3.34), it is sufficient to satisfy

$$\overline{F} + (\lambda + \bar{\lambda})F - \rho = -\frac{\alpha}{\sqrt{2}}$$

that is

$$\rho = \overline{F} + (\lambda + \bar{\lambda})F + \frac{\alpha}{\sqrt{2}}$$

So, the control ν which drives s to zero in finite time

$$t_{reach} \leq \frac{\sqrt{2}}{\alpha}|s(0)|$$

corresponds to

$$\nu = -\bar{\lambda}\lambda x_2 - (\lambda + \bar{\lambda})u - \rho\,\text{sign}(s)$$

It is worthwhile to notice that, since ν contains the sign function, it will be affected by the chattering phenomena. However, since the actual command u is the integral of ν, such issue will be greatly reduced.

Notice that, to compute the command ν one must know s, which requires the computation of $\dot{\sigma}$. To this purpose, numerical solutions can be exploited.