Cart-Pendulum Mathematical Model (draft version)

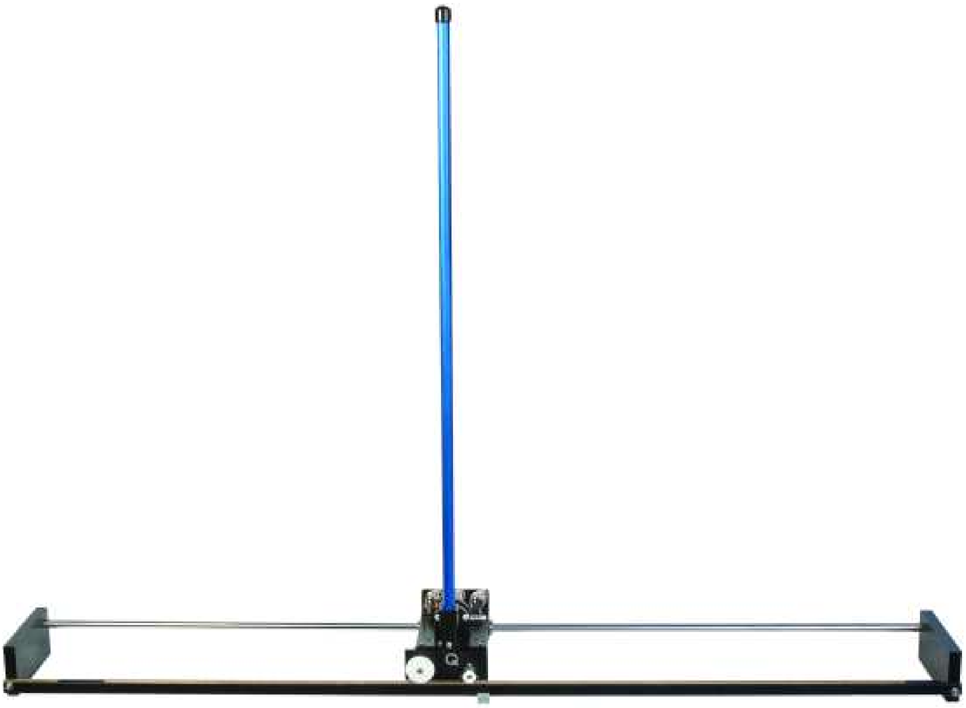

Figure 1: The Quanser inverted pendulum IP02

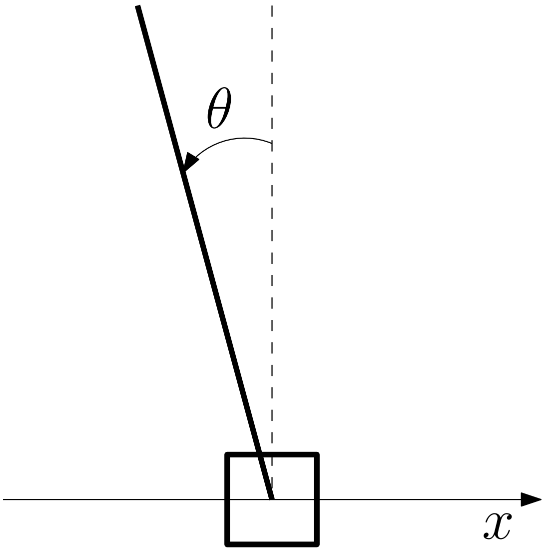

Figure 2: Sketch of the inverted pendulum

Parameters

- M: mass of the cart.
- m: mass of the pendulum.
- L: length of the pendulum.
- l: distance from the pivot to the center of mass ($l = L/2$).
- I: moment of inertia of the pendulum ($I = \frac{4}{3}ml^2$).
- x: horizontal position of the cart.
- θ: angle of the pendulum (measured in the opposite direction to the standard).
- F: external force applied to the cart.
- b_x: viscous friction coefficient of the cart.
- b_θ: viscous friction coefficient of the pendulum pivot.
- k_a: overall gain of the actuator

Nonlinear Model

Let the command $u \in [-5, 5]$ be the voltage applied to the motor mounted on the cart.
The nonlinear equations of motion are:

$$\begin{cases} (M+m)\ddot{x} = k_a u - b_x \dot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2 \sin\theta & (1) \\ I\ddot{\theta} = -b_\theta \dot{\theta} + ml\ddot{x}\cos\theta + mgl\sin\theta & (2) \end{cases}$$

Let us write the model by solving for $\ddot{x}$ and $\ddot{\theta}$, obtaining

$$\begin{cases} \ddot{x} = \dfrac{I(k_a u - b_x\dot{x} - ml\dot{\theta}^2\sin\theta) + ml\cos\theta(mgl\sin\theta - b_\theta\dot{\theta})}{I(M+m) - (ml\cos\theta)^2} & (3) \\ \\ \ddot{\theta} = \dfrac{(M+m)(mgl\sin\theta - b_\theta\dot{\theta}) + ml\cos\theta(k_a u - b_x\dot{x} - ml\dot{\theta}^2\sin\theta)}{I(M+m) - (ml\cos\theta)^2} & (4) \end{cases}$$

Linearized State-Space Model

Assuming small oscillations around the unstable equilibrium point ($\theta \approx 0,\ \dot{\theta} \approx 0$), we can apply the following approximations:

- $\sin\theta \approx \theta$
- $\cos\theta \approx 1$
- $\dot{\theta}^2 \approx 0$

The linearized system of equations becomes:

$$\begin{cases} (M+m)\ddot{x} + b_x\dot{x} - ml\ddot{\theta} = k_a u \\ I\ddot{\theta} + b_\theta\dot{\theta} - ml\ddot{x} - mgl\theta = 0 \end{cases} \tag{5}$$

Define the state vector $\mathbf{z} = [x, \dot{x}, \theta, \dot{\theta}]^T$ and assume the outputs of interest are the cart position x and the pendulum angle θ. The system can be written in the form

$$\begin{cases} \dot{\mathbf{z}} & = A\mathbf{z} + Bu \\ \mathbf{y} & = C\mathbf{z} \end{cases} \tag{6}$$

Let $p = I(M+m) - (ml)^2$. The matrices A, B, C, D are:

$$A = \frac{1}{p}\begin{bmatrix} 0 & p & 0 & 0 \\ 0 & -Ib_x & m^2gl^2 & -mlb_\theta \\ 0 & 0 & 0 & p \\ 0 & -mlb_x & mgl(M+m) & -(M+m)b_\theta \end{bmatrix}, \quad B = \frac{k_a}{p}\begin{bmatrix} 0 \\ I \\ 0 \\ ml \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Observer design

Full-order asymptotic observer

The Luenberger asymptotic observer is governed by

$$\begin{cases} \dot{\widehat{\mathbf{z}}} & = A\,\widehat{\mathbf{z}} + Bu + L(\mathbf{y} - \widehat{\mathbf{y}}) \\ \widehat{\mathbf{y}} & = C\,\widehat{\mathbf{z}} \end{cases} \tag{7}$$

Let the error be defined as $\mathbf{e} = \mathbf{z} - \widehat{\mathbf{z}}$, it is easy to see that

$$\dot{\mathbf{e}} = (A - LC)\mathbf{e}$$

So, the matrix L must be designed to guarantee $(A - LC)$ asymptotically stable with chosen poles.

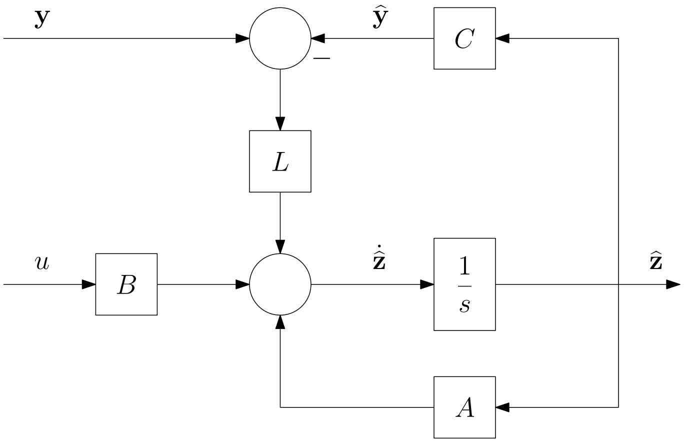

Figure 3: Full-order asymptotic observer

Controller design

Several types of controllers (both linear and nonlinear) can be designed to control the pendulum around the unstable equilibrium point. In our implementation, a Linear Quadratic Regulator (LQR) with extended states is implemented.

Swing-up controller

A possible solution to design the swing-up controller is based on the energy of the pendulum. The idea is to increase it until swing occurs.
The potential energy (with respect to the downward pendulum) is

$$E_p = mgl(1+\cos(\theta))$$

while the kinetic energy is

$$E_k = \frac{1}{2} I\dot{\theta}^2$$

where $I = \frac{4}{3}ml^2$ denote the moment of inertia of the pendulum.
Then, the total energy of the pendulum is

$$E = \frac{1}{2} I\dot{\theta}^2 + mgl(1+\cos(\theta))$$

Taking its time derivative one has

$$\dot{E} = I\dot{\theta}\ddot{\theta} - \dot{\theta}mgl\sin\theta \tag{8}$$

From (2) one has

$$mgl\sin\theta = I\ddot{\theta} + b_\theta\dot{\theta} - ml\ddot{x}\cos\theta$$

Substituting in (8), one gets

$$\dot{E} = -b_\theta\dot{\theta}^2 + ml\dot{\theta}\ddot{x}\cos\theta \tag{9}$$

Notice that, besides of θ and $\dot{\theta}$, the derivative of the energy depends on the cart acceleration $\ddot{x}$. The aim is to choose the command u such that $\ddot{x}$ such that the term $ml\dot{\theta}\ddot{x}\cos\theta$ is positive. This is obtained when

$$sign(\ddot{x}) = sign(\dot{\theta}\cos\theta)$$

From (3),

$$\ddot{x} = \underbrace{\frac{Ik_a}{I(M+m)-(ml\cos\theta)^2}}_{\eta} u + \Gamma = \eta u + \Gamma \tag{10}$$

where Γ is a function which does not depend on u. It can be proved that the denominator in (10) is always positive. Since both I and k_a are positive, one has $\eta > 0$. By substituting in (11) one gets

$$\dot{E} = -b_\theta\dot{\theta}^2 + \Gamma ml\dot{\theta}\cos\theta + \eta ml(\dot{\theta}\cos\theta)\,u \tag{11}$$

Notice that the first term in (11) is a dissipative term due to friction, and it is always negative. Moreover, since the friction b_θ is small, such a term is negligible. The sign of the second term changes depending on θ and $\dot{\theta}$; however, it does not depend directly on u, and so we cannot directly change it. So, we aim at maximizing the third term. By choosing it as

$$u = \mu \cdot sign(\dot{\theta}\cos\theta) \tag{12}$$

the third term becomes

$$\eta ml\mu|\dot{\theta}\cos\theta|$$

which is positive for any $\mu > 0$.
It remains to choose μ in (12). Let *sat* denotes the command saturation limit. Several approaches can be taken. For instance:

- $\mu = \overline{M}$, where $\overline{M} \gg sat$. The control command becomes

$$u \simeq \begin{cases} -sat & \text{if } sign(\dot{\theta}\cos\theta) < 0 \\ sat & \text{if } sign(\dot{\theta}\cos\theta) > 0 \end{cases}$$

- $\mu(t) = k_q(E_{up} - E(t))$, where $k_q > 0$, $E_{up} = 2mgl$ is the pendulum energy at upward position and $E(t)$ is the pendulum energy at the current time t. In this case the command is proportional to the energy gap between E_{up} and $E(t)$.