Chapter 1

Absolute stability

1.1 Analysis of absolute stability (Lure system)

Let us consider systems which can be written as a linear strictly proper block $G(s)$ in the forward path and a static nonlinear block $\Psi(t, y)$ in the feedback path.

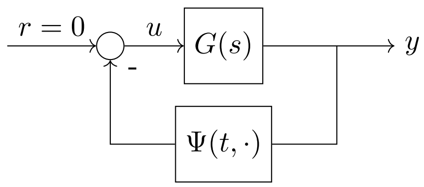


Figure 1.1: General scheme of a Lure system.

Many systems can be represented in this way, like e.g., systems where the nonlinearity affects the actuator. Let us describe the linear function in its state-space form, i.e.,

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \\ u = -\Psi(t, y) \end{cases} \tag{1.1}$$

where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^p$. Moreover, we assume that the reference input is null, i.e., $r = 0$ and that the couple (A, B) is controllable and (A, C) is observable.

The block $\Psi(\cdot, \cdot) : [0, \infty) \times \mathbb{R}^p \to \mathbb{R}^p$ is a memoryless (static) nonlinearity, possibly time-varying, piecewise continuous in t and locally Lipschitz in y. Such nonlinearity must satisfy a *sector condition* as explained below.

Let us analyze the sector condition in the scalar case, i.e., $p = 1$. The function $\Psi(\cdot, \cdot) : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ satisfies the sector condition if there exist constants α, β, a, b such that

$$\begin{aligned} &\alpha y^2 \le y\Psi(t, y) \le \beta y^2 \,,\ \forall\, t \ge 0, \forall\, y \in [a, b] \\ &\text{subject to } \alpha < \beta \text{ and } a < 0 < b \end{aligned} \tag{1.2}$$

The sector condition may hold in an interval $[a, b]$ or over the whole domain $y \in (-\infty, \infty)$; in the latter case, we say that the sector condition holds globally. An example is depicted in Fig. 1.2.

An equivalent formulation of (1.2) is as follows.

$$[\Psi(t, y) - \alpha y][\Psi(t, y) - \beta y] \le 0, \forall\, t \ge 0, \forall y \in [a, b] \tag{1.3}$$

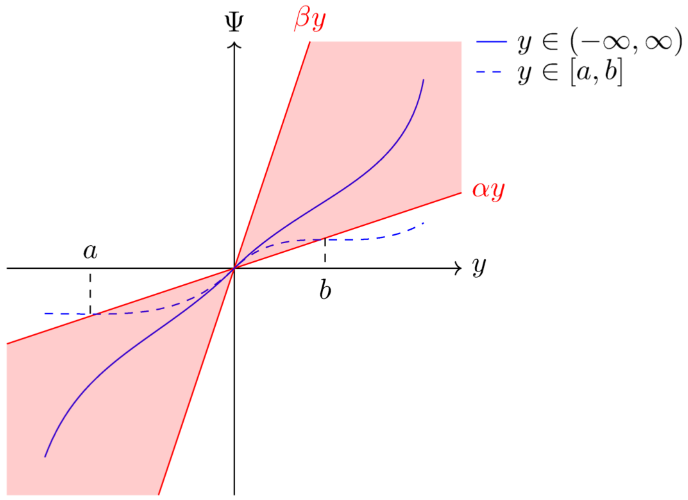


Figure 1.2: Graphical representation of nonlinearities satisfying the sector condition (case: $\beta > \alpha > 0$).

It is easy to show this statement. In fact, by (1.2) one has

$$\begin{cases} \alpha y \leq \Psi(t,y) \leq \beta y\,, & \text{if } y \geq 0 \\ \beta y \leq \Psi(t,y) \leq \alpha y\,, & \text{if } y \leq 0 \end{cases}$$

which clearly implies (1.3). The systems which satisfy this condition are called Lure systems.

Let us generalize these results to the MIMO case.

$$\Psi(t,y) = \left.\begin{bmatrix} \Psi_1(t,y_1) \\ \Psi_2(t,y_2) \\ \vdots \\ \Psi_p(t,y_p) \end{bmatrix} \begin{matrix} \to \\ \to \\ \vdots \\ \to \end{matrix} \right\} [\alpha_1, \alpha_2, \cdots, \alpha_p], [\beta_1, \beta_2, \cdots, \beta_p] : \alpha_i < \beta_i,\ i = 1, \ldots, p$$

Now, we have p independent channels and we suppose that each component satisfies the sector condition defined by α_i, β_i. Let us define the orthotope

$$\Gamma = \{y \in \mathbb{R}^p : a_i \leq y_i \leq b_i, \quad i = 1, \ldots, p\}$$

and the diagonal matrices

$$K_{\min} = \text{diag}\{\alpha_1, \alpha_2, \cdots, \alpha_p\}$$
$$K_{\max} = \text{diag}\{\beta_1, \beta_2, \cdots, \beta_p\}$$

So, we can write the sector condition (1.3) for MIMO systems as

$$[\Psi(t,y) - K_{\min} y]^T [\Psi(t,y) - K_{\max} y] \leq 0,\ \forall t,\ \forall y \in \Gamma \tag{1.4}$$

Notice that this is a special case, because we suppose that each Ψ_i is independent. In general, $K_{\min}$ and $K_{\max}$ are matrices such that:

$$K = K_{\max} - K_{\min} > 0$$

with K symmetric and positive defined.

We may summarize the previous reasoning in the following definition.

Definition 1.1. *A memoryless nonlinearity* $\psi\colon [0,\infty)\times\mathbb{R}^p \to \mathbb{R}^p$ *satisfies a sector condition if*

$$[\Psi(t,y) - K_{\min}y]^T[\Psi(t,y) - K_{\max}y] \le 0,\ \forall t,\ \forall y \in \Gamma \tag{1.5}$$

for some real matrices $K_{\min}$*,* $K_{\max}$ *such that* $K_{\max} - K_{\min}$ *is a positive definite symmetric matrix and the interior of* Γ *is connected and contains the origin.*

Example 1.1. The concept of sector condition can be applied to more general nonlinearities. Let us suppose the following inequality holds:

$$\|\Psi(t,y) - Ly\|_2 \le \gamma\|y\|_2 \quad ,\ \gamma > 0 \tag{1.6}$$

We want to rewrite (1.6) as a sector condition. Let us square and take the difference. One has

$$\|\Psi(t,y) - Ly\|_2^2 - \gamma^2\|y\|_2^2 \le 0$$

Now, we can rewrite this inequality to obtain a form which is similar to the generalized sector condition in (1.5).

$$\begin{aligned}
&[\Psi - Ly]^T[\Psi - Ly] - \gamma^2 y^T y \le 0\\
&[\Psi - Ly]^T[\Psi - Ly] - \gamma^2 y^T y + \gamma[\Psi - Ly]^T y - \gamma[\Psi - Ly]^T y \le 0\\
&[(\Psi - Ly) + \gamma y]^T[(\Psi - Ly) - \gamma y] \le 0\\
&[\Psi - \underbrace{(L - \gamma I)}_{K_{\min}} y]^T[\Psi - \underbrace{(L + \gamma I)}_{K_{\max}} y] \le 0
\end{aligned}$$

So, condition (1.6) can be stated as the following sector condition

$$[\Psi - K_{\min}y]^T[\Psi - K_{\max}y] \le 0$$

where $K_{max} - K_{min}$ is a positive definite (diagonal) matrix. So, if the nonlinear function satisfies the sector condition, (1.6) holds. △

By (1.1), it is easy to see that the origin $x = 0$ is an equilibrium point. The aim of this chapter is to study if the origin is stable or not for all the possible functions satisfying the sector condition. If the origin is asymptotically stable for all the nonlinearities satisfying a sector condition, then the system is said *absolutely stable.*

To study the absolute stability of the origin, we will use a Lyapunov approach. In particular, let us consider the following Lyapunov function[1]:

$$V(x) = x^T P x,\ P = P^T > 0$$

Let $\dot{x} = f(x)$, if the condition below holds, then the function is globally asymptotically stable (see Fig. 1.3).

$$\dot{V}(x) = \underbrace{\left(\frac{\partial V}{\partial x}\right)^T}_{gradient} \cdot f(x) < 0\,, \quad \forall\, x \ne 0$$

If this condition does not hold, nothing can be concluded.

[1] A function $V(x)\colon \mathbb{R}^n \to \mathbb{R}$ is said a Lyapunov function if $V(0) = 0$ and $V(x) > 0,\ \forall x \ne 0$.

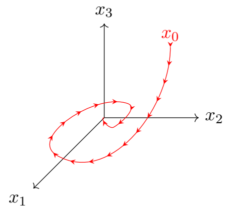


Figure 1.3: Sketch of global asymptotic stability in $\mathbb{R}^3$.

Consider a different Lyapunov function, called Lure-type Lyapunov function

$$V(x) = x^T P x + 2\eta \int_0^y \Psi^T(\sigma) K \, d\sigma$$

with $\eta \in \mathbb{R}_+$, $K > 0$ positive definite symmetric matrix and Ψ time invariant.

We will have different criteria based on the two different Lyapunov functions.

$$V(x) = x^T P x \qquad \text{Circle Criterion}$$

$$V(x) = x^T P x + 2\eta \int_0^y \Psi^T(\sigma) K d\sigma \qquad \text{Popov Criterion}$$

To analyze the absolute stability, we must introduce the concept of positive-real transfer functions. We will find frequency-domain conditions on $G(s)$ and K such that $\dot{V} < 0$.

1.2 Positive-real transfer functions

Let us consider a square, proper, rational, t.f.m. $Z(s) = \{z_{ij}(s)\} \in \mathbb{C}^{p \times p}$.

Definition 1.2 (Positive-Real Transfer Function (PR)). *A transfer function matrix $Z(s)$ is PR if:*

1. *all the poles of $z_{ij}(s) \notin RHP$;*

2. *any pole of $z_{ij}(s)$ on the imaginary axis is simple and the associated residue matrix of $Z(s)$ is positive semidefinite Hermitian;*

3. *for any ω such that $j\omega$ is not a pole of any $z_{ij}(s)$, one has $Z(j\omega) + Z^T(-j\omega) \geq 0$.*

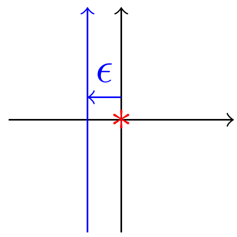
Definition 1.3 (Strictly Positive-Real Transfer Function (SPR)). *A transfer function matrix $Z(s)$ is SPR if:*

- $\exists \epsilon > 0 : Z(s - \epsilon)$ *is PR*

Remark. Notice that, for SISO systems ($p = 1$), Condition 3 in Definition 1.2 becomes $Re[Z(j\omega)] \geq 0$, i.e., the Nyquist plot of $Z(j\omega)$ lies in the CRHP.

Remark. By Condition 3 in Definition 1.2, and by the previous remark, any scalar function with relative degree > 1 cannot be PR.

Example 1.2. For instance, $G(s) = \frac{1}{s}$ is PR but not SPR. In fact, $G(s) = \frac{1}{s-\epsilon}$ is not PR for any $\epsilon > 0$ since it has a pole in RHP.



△

Let $\mathcal{H}$ denotes the space of Hurwitz transfer functions, i.e., the space of all transfer functions which have all their poles in the open left half plane, i.e., no poles in CRHP. Sometimes to assess SPR it is convenient to make use of the following lemma.

Lemma 1.1. *Assume* $Z(s) \in \mathbb{C}^{p\times p}$ *and suppose* $\det[Z(s)+Z^T(-s)]$ *is not identically 0, i.e. it has normal rank p. Then* $Z(s)$ *is SPR if and only if:*

- $Z(s) \in \mathcal{H}$
- $Z(j\omega) + Z^T(-j\omega) > 0,\ \forall \omega \in \mathbb{R}$
- *one of the following conditions holds:*
  1. $Z(\infty) + Z^T(\infty) > 0$
  2. $Z(\infty) + Z^T(\infty) = 0$, *and* $\lim_{\omega\to\infty} \omega^2[Z(j\omega) + Z^T(-j\omega)] > 0$
  3. $Z(\infty)+Z^T(\infty) \geq 0$, *and* $\exists \sigma_0, \omega_0 > 0 \colon \omega^2\underline{\sigma}[Z(j\omega)+Z^T(-j\omega)] \geq \sigma_0,\ \forall |\omega| \geq \omega_0$

Example 1.3. Let us report some examples of PR and SPR functions.

- $Z(s) = \dfrac{1}{s+a},\ a > 0$ is SPR.
- $Z(s) = \dfrac{1}{s^2+s+1}$ is not PR.
- $Z(s) = \dfrac{1}{s+1}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Since $\det[Z(s) + Z(-s)^T] = 0$, we cannot apply Lemma 1.1. However, by Definitions 1.2-1.3, we may conclude that $Z(s)$ is SPR.
- $Z(s) = \dfrac{1}{s+1}\begin{bmatrix} s+1 & 1 \\ -1 & 2s+1 \end{bmatrix}$ is SPR by Lemma 1.1, condition 1.
- $Z(s) = \begin{bmatrix} \dfrac{s+2}{s+1} & \dfrac{1}{s+2} \\ \dfrac{-1}{s+2} & \dfrac{2}{s+1} \end{bmatrix}$ is SPR by Lemma 1.1, condition 2.

△

The following lemma states some conditions to assess SPR.

Lemma 1.2 (Kalman-Yakubovich-Popov (KYP) or Positive Real)**.** *Let*

$$Z(s) = \mathcal{C}(sI - \mathcal{A})^{-1}\mathcal{B} + \mathcal{D}$$

with $Z(s) \in \mathbb{C}^{p\times p}$ *and* $\mathcal{A} \in \mathcal{H}$ *(Hurwitz),* $(\mathcal{A}, \mathcal{B})$ *controllable,* $(\mathcal{A}, \mathcal{C})$ *observable.* $Z(s)$ *is SPR if and only if* $\exists P = P^T > 0$*,* L, W*, and* $\epsilon \in \mathbb{R}$*:* $\epsilon > 0$ *such that:*

- $P\mathcal{A} + \mathcal{A}^T P = -L^T L - \epsilon P$
- $P\mathcal{B} = \mathcal{C}^T - L^T W$
- $\mathcal{D} + \mathcal{D}^T = W^T W$

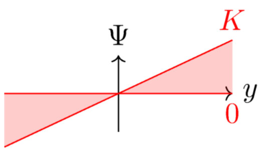
The KYP Lemma will be useful since it links the SPR property to the existence of a suitable Lyapunov function.

1.3 Circle criterion

Let us consider an asymptotically stable system ($A \in \mathcal{H}$, (A,B) controllable, (A,C) observable). Assume to have a memoryless nonlinearity in the feedback loop, as in Fig. 1.1.

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \\ u = -\Psi(t, y) \end{cases} \rightarrow \begin{cases} \dot{x} = Ax - B\Psi(t, y) \\ y = Cx \end{cases}$$

Assume that the nonlinearity satisfies a sector condition that has the lower bound equal to 0, i.e., $K_{min} = 0$. Moreover, assume that $K_{max} = K$ is symmetric and positive definite ($K > 0$).



So, condition (1.5) becomes

$$\Psi^T(\Psi - Ky) \leq 0 \quad \rightarrow \quad \Psi^T(\Psi - KCx) \leq 0$$

Let us consider the Lyapunov function $V(x) = x^T Px$, $P = P^T > 0$ and let us derive it.

$$\dot{V}(x) = \left(\frac{\partial V}{\partial x}\right)^T \dot{x} = (2x^T P)(Ax - B\Psi) = x^T(2PA)x - 2x^T PB\Psi \tag{1.7}$$

Note. Given a generic square matrix V, it can always be written as $V = V_s + V_a$, where V_s and V_a denote the symmetric and anti-symmetric part, defined as $V_s = (V + V^T)/2$ and $V_a = (V - V^T)/2$. It is easy to see that, in a quadratic form, the anti-symmetric part does not provide any contribution and then $X^T V X = X^T V_s X = X^T[(V + V^T)/2]X$.

So, we may write (1.7) as

$$\dot{V}(x) = x^T(PA + A^T P)x - 2x^T PB\Psi$$

Now, we subtract the sector condition:

$$\begin{aligned}\dot V(x) &\le x^T(PA + A^TP)x - 2x^TPB\Psi - 2\Psi^T(\Psi - KCx)\\ &= x^T(PA + A^TP)x + 2x^T[C^TK - PB]\Psi - 2\Psi^T\Psi\end{aligned}$$

We want to show that the right-hand term is negative. Suppose that $\exists\, P = P^T > 0, L, \epsilon > 0$ such that

$$\begin{cases} PA + A^TP = -L^TL - \epsilon P \\ PB = C^TK - \sqrt{2}L^T \end{cases} \tag{1.8}$$

Then,

$$\begin{aligned}\dot V(x) &\le x^T(PA + A^TP)x + 2x^T[C^TK - PB]\Psi - 2\Psi^T\Psi = \\ &= -\epsilon x^TPx - \underbrace{(x^TL^TLx - 2\sqrt{2}x^TL^T\Psi + 2\Psi^T\Psi)}_{[\sqrt{2}\Psi - Lx]^T[\sqrt{2}\Psi - Lx]} = \\ &= -\epsilon x^TPx - \underbrace{[\sqrt{2}\Psi - Lx]^T[\sqrt{2}\Psi - Lx]}_{\ge 0} \\ &\le -\epsilon x^TPx < 0\end{aligned}$$

This means that the derivative is strictly negative (under the above assumptions), which means that the system is asymptotically stable. It remains to prove that $\exists\, P = P^T > 0, L, \epsilon > 0$ exist such that (1.8) holds.

Since (A, C) is observable and K is full rank, than also (A, KC) is observable. Using the KYP Lemma, we can construct

$$\begin{cases} \mathcal{A} = A \\ \mathcal{B} = B \\ \mathcal{C} = KC \\ W = \sqrt{2}I \;\rightarrow\; W^TW = 2I \;\rightarrow\; \mathcal{D} = I \end{cases} \qquad \Rightarrow Z(s) = I + KC(sI - A)^{-1}B$$

So, if $Z(s)$ is SPR, then (1.8) holds and $\dot V < 0$ which implies that the system is absolutely stable. Summarizing, if we consider $G(s) = C(sI - A)^{-1}B$, then

$$Z(s) = I + KG(s) \text{ is SPR} \implies \text{absolute stability}$$

We may summarize the above results in the following lemma.

Lemma 1.3. *Let the system* (1.1) *be given, where $A \in \mathcal{H}$, (A, B) is controllable and (A, C) is observable. Let Ψ satisfy the sector condition $\Psi^T(\Psi - Ky) \le 0$. Then, the system is absolutely stable if $Z(s) = I + KG(s)$ is SPR.*

Note. This condition is a sufficient condition but not necessary. So, Lemma 1.3 gives only positive answer to the absolute stability problem.

1.3.1 Loop transformation (pole shifting)

If A is not Hurwitz or the nonlinearity does not satisfy $\Psi^T(\Psi - Ky) \le 0$, Lemma (1.3) cannot be applied. A possible solution to overcome this problem is given by the so-called *loop transformation* or *pole shifting* procedure (see Fig. 1.4).

As an exercise, one may show that the transformed system in Fig. 1.4 is equivalent to the original one in Fig. 1.1. In practice, we rotate the original sector by $K_{\min}y$

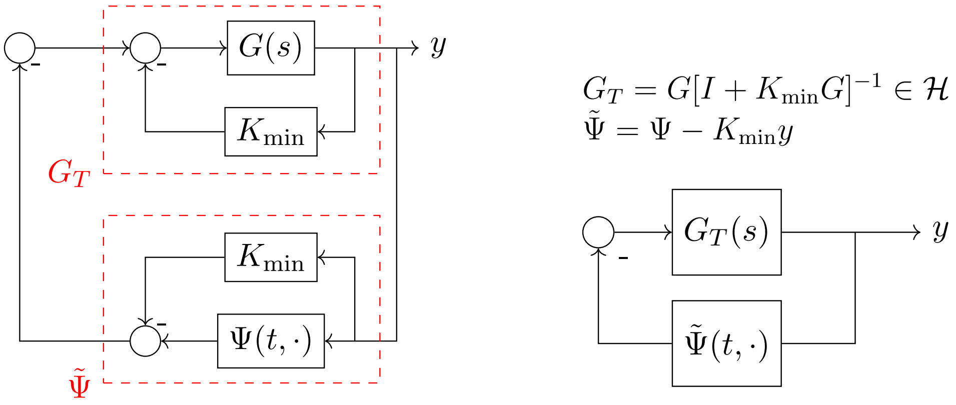
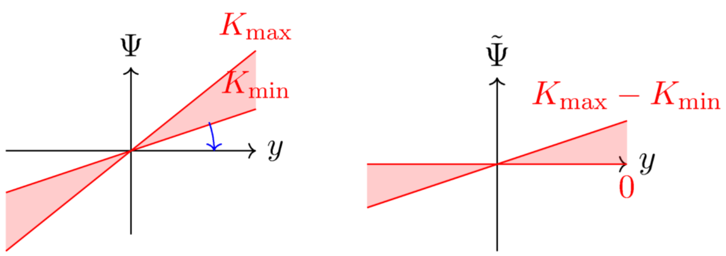


Figure 1.4: Pole shifting procedure.

in order to obtain a new sector which satisfies the condition of Lemma 1.3, i.e., where the new nonlinearity $\tilde{\Psi}$ is bounded in $[0, (K_{\max} - K_{\min})y] = [0, Ky]$. This procedure leads to a new system on which we can apply Lemma 1.3, provided that G_T be Hurwitz. In the following illustrations, we can visualize the procedure graphically.



By the circle criterion (Lemma 1.3), the sufficient conditions for absolute stability of Lure systems are

- $G_T = G[I + K_{\min}G]^{-1} \in \mathcal{H}$
- $Z_T = I + KG_T = I + (K_{\max} - K_{\min})G_T$ is SPR

We may rewrite the second condition as

$$\begin{aligned} Z_T &= [I + K_{\min}G][I + K_{\min}G]^{-1} + (K_{\max} - K_{\min})G[I + K_{\min}G]^{-1} = \\ &= [I + K_{\max}G][I + K_{\min}G]^{-1} \text{ is SPR} \end{aligned}$$

The overall procedure is summarized in the following theorem.

Theorem 1.1 (Multivariable circle criterion)**.** *Let the system* (1.1) *be given, where* (A, B) *is controllable,* (A, C) *is observable, and* Ψ *satisfies the sector condition* (1.5)*. Then, the system is absolutely stable if*

- $G_T = G[I + K_{\min}G]^{-1} \in \mathcal{H}$
- $Z_T = [I + K_{\max}G][I + K_{\min}G]^{-1}$ *is SPR*

Notice that, if $G(s) \in \mathcal{H}$ and $K_{min} = 0$, Theorem 1.1 reduces to Lemma 1.3. Moreover, since $Z_T(\infty) = I$, by Lemma 1.1 follows that Z_T if SPR if and only if Z_T is Hurwitz and

$$Z_T(j\omega) + Z_T^T(-j\omega) > 0, \; \forall \omega \in \mathbb{R}$$

1.3.2 Circle criterion for SISO systems

Let us consider a SISO system and let us refer to the scheme in Fig. 1.1. Matrices $K_{\min}$ and $K_{\max}$ can be replaced by α and β, respectively. According to Theorem 1.1, the conditions for absolute stability become:

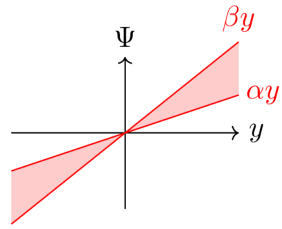
1. $G_T = \dfrac{G}{1+\alpha G} \in \mathcal{H}$
2. $Z_T = \dfrac{1+\beta G}{1+\alpha G}$ is SPR

To guarantee the second condition, we require $Z_T \in \mathcal{H}$ and

$$\mathrm{Re}[Z_T(j\omega)] = \mathrm{Re}\left[\frac{1+\beta G(j\omega)}{1+\alpha G(j\omega)}\right] > 0,\ \forall \omega \in \mathbb{R} \tag{1.9}$$

Depending on the shape of the sector condition, we have to consider 3 cases: $(\beta > \alpha > 0)$, $(\beta > \alpha = 0)$ and $(\beta > 0 > \alpha)$.

1. **Case $\boldsymbol{\beta > \alpha > 0}$**: In this case, the corresponding sector of Ψ is of the following form



Since α and β are both positive, condition (1.9) can be rewritten as

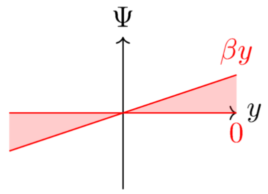
$$\mathrm{Re}\left[\frac{\frac{1}{\beta}+G(j\omega)}{\frac{1}{\alpha}+G(j\omega)}\right] > 0 \tag{1.10}$$

Looking at the Nyquist plot in Fig. 1.5, one may prove that to satisfy (1.10) it suffices that the angle γ belongs to $(-\frac{\pi}{2}, \frac{\pi}{2})$, $\forall\omega$. So, we need $\gamma = (\gamma_1 - \gamma_2) \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $\forall\omega$. In fact, the real part of the ratio of two complex numbers is positive if and only if the phase difference between the two numbers belongs to $(-\frac{\pi}{2}, \frac{\pi}{2})$. It can be proven that this condition holds for any point outside the red circle in Fig. 1.5. Let us denote such a circle as $\mathcal{C}\left(-\frac{1}{\alpha}, -\frac{1}{\beta}\right)$.

Moreover, G_T is Hurwitz if and only if $G(j\omega)$ does not cross the point $-1/\alpha$ and encircles it m times counterclockwise, where m denotes the number of poles of G in the RHP. Now, we may state the following theorem.

Theorem 1.2 (Circle criterion SISO, case $\beta > \alpha > 0$)**.** *Let $\beta > \alpha > 0$. The SISO system (1.1) is absolutely stable if the Nyquist diagram of $G(j\omega)$ encircles the circle $\mathcal{C}(-1/\alpha, -1/\beta)$ m times counterclockwise, where m denotes the number of poles of G in RHP.*

2. **Case $\boldsymbol{\beta > \alpha = 0}$**: In this case, Ψ must lie in the following sector



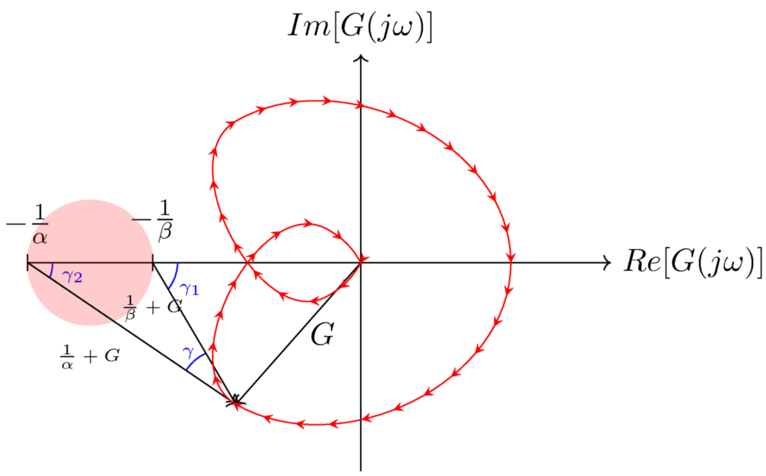


Figure 1.5: Circle criterion for SISO systems in case $\beta > \alpha > 0$.

So, we can imagine that $-\frac{1}{\alpha} \to -\infty$ for $\alpha \to 0$, and then the radius of the forbidden circle goes to infinite. In practice, the circle $\mathcal{C}(-1/\alpha, -1/\beta)$ collapses to the half-space $\mathrm{Re}[G(j\omega)] \leq -\frac{1}{\beta}$, see Fig. 1.6.

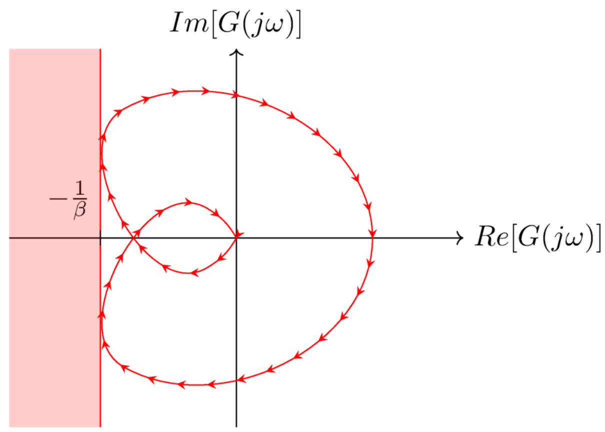
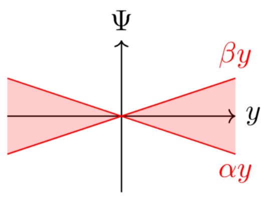


Figure 1.6: Circle criterion for SISO systems in case $\beta > \alpha = 0$.

Theorem 1.3 (Circle criterion SISO, case $\beta > \alpha = 0$)**.** *Let* $\beta > \alpha = 0$. *The SISO system* (1.1) *is absolutely stable if* $G \in \mathcal{H}$ *and* $Re[G(j\omega)] > -\frac{1}{\beta}$, $\forall \omega \in \mathbb{R}$.

Notice that, this case encloses $\Psi(y) = 0$, which implies that the open loop function $G(s)$ be stable.

3. **Case $\boldsymbol{\beta > 0 > \alpha}$**: In this case, the correspondent sector of Ψ is of the following form



Also in this case, the sector contains the real axis which implies that $G(s)$ must be asymptotically stable. Moreover, the fact that α is negative inverts the relative position of $-\frac{1}{\beta}$ and $-\frac{1}{\alpha}$ which leads the forbidden zone to be the complementary region of the first case, see Fig. 1.7.

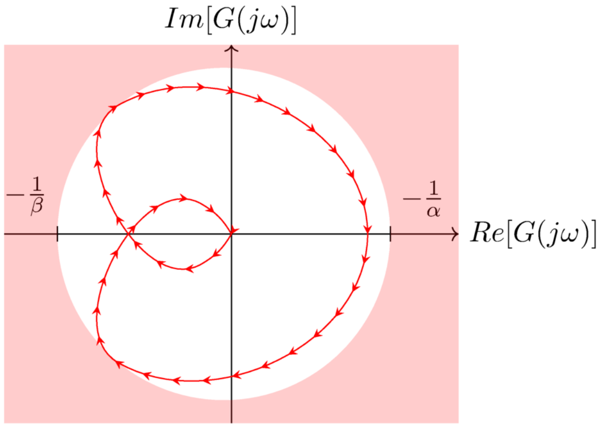


Figure 1.7: Circle criterion for SISO systems in case $\beta > 0 > \alpha$.

The following theorem states the condition for absolute stability in this case.

Theorem 1.4 (Circle criterion SISO, case $\beta > 0 > \alpha$)**.** *Let $\beta > 0 > \alpha$. The SISO system* (1.1) *is absolutely stable if $G \in \mathcal{H}$ and the entire Nyquist diagram of $G(j\omega)$ lies inside the circle $\mathcal{C}(-1/\beta, -1/\alpha)$.*

Example 1.4. Let us consider the following linear system

$$G(s) = \frac{4}{(1+s)(1+\frac{s}{2})(1+\frac{s}{3})} \in \mathcal{H}$$

The Nyquist plot is depicted in Fig. 1.8.

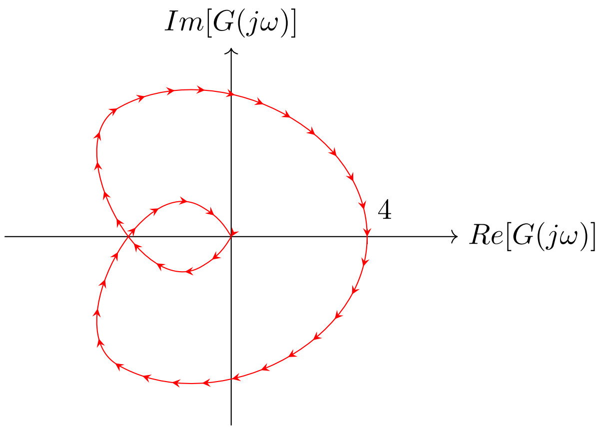


Figure 1.8: Nyquist plot of $G(j\omega)$ in Example 1.4.

We want to find under which sector conditions of the nonlinearity Ψ the overall system is absolutely stable. Let us start looking to the third case, that is $\alpha < 0$ and $\beta > 0$. What we are going to do is similar to the situation depicted in Fig. 1.7. We can take for instance $-\frac{1}{\alpha} = 4$ and $-\frac{1}{\beta} = -4$ choosing the origin as the center of the admissible circular zone $\mathcal{C}(-4, 4)$. This procedure leads to an admissible but very narrow sector. In fact, the nonlinearity must satisfy strict conditions, where $\alpha = -0.25$ and $\beta = 0.25$. Therefore, we try to extend the sector of Ψ by choosing a different center of the circular zone and a different radius in order to enlarges the sector. For example, we can choose as center the point on the real axis that is inside the Nyquist plot and has the greatest distance from it, and as radius this distance. The computation of this values gives as result the point 1.5 as center and 2.9 as radius, i.e., the circle $\mathcal{C}(-1.4, 4.4)$. The corresponding sector is larger than before, being $\alpha = -0.23$ and $\beta = 0.71$.

Now, let us assume that the nonlinearity Ψ is a saturation (see Fig. 1.9). To study this case, we may impose $\alpha = 0$. So, we can use the tangent on the left plane to the Nyquist plot and enlarge the section to $\alpha = 0$, $\beta = 1.17$. In other words, we choose as $-\frac{1}{\beta}$ the $\min_\omega \{\text{Re}[G(j\omega)]\}$. This situation is similar to the one depicted in Fig. 1.6. △

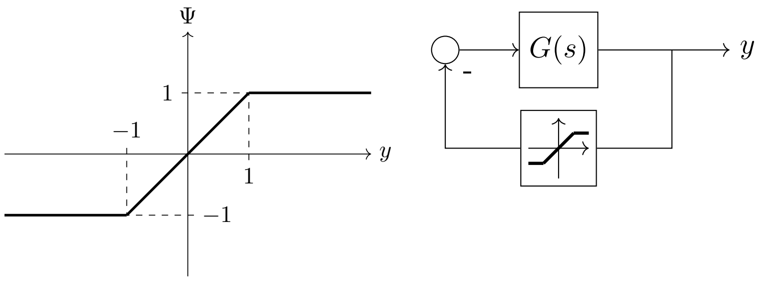


Figure 1.9: Example 1.4. The nonlinearity Ψ is a saturation.

Example 1.5. Assume that

$$G(s) = \frac{4}{(s-1)(1+\frac{s}{2})(1+\frac{s}{3})} \notin \mathcal{H}$$

Since the transfer function is unstable, we must consider the first case, which implies $\beta > \alpha > 0$. To achieve absolute stability, we must impose that the Nyquist plot encircles counterclockwise the forbidden disc (without crossing it) a number of times equal to the number of unstable poles, in this case once. We obtain that a possible center is -3.5, with radius 0.185 leading to the $\alpha = 0.27$ and $\beta = 0.30$ (see Fig. 1.10). In this case, we obtain a very narrow band because we aim at stabilizing an unstable system affected by a sector nonlinearity. △

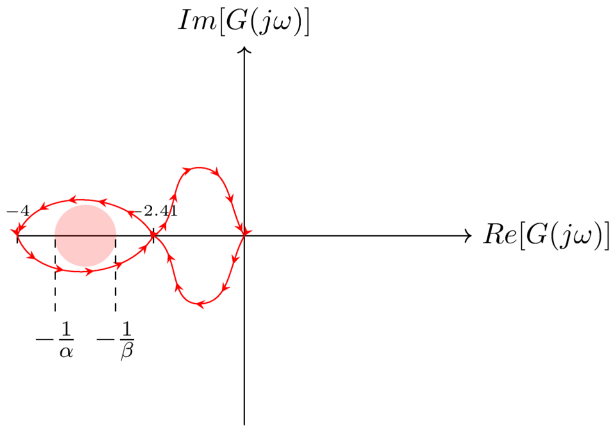


Figure 1.10: Example 1.5. Application of the circle criterion.

Example 1.6. Let

$$G(s) = \frac{s+2}{(s+1)(s-1)}$$

and assume that the nonlinearity be a saturation as in Fig. 1.9. In this case we cannot guarantee the absolute stability in the whole domain. In fact, since $G \notin \mathcal{H}$, to have absolute stability we must refer to the first case of the circle criterion, which enforces $\alpha > 0$. It is clear that having $\alpha > 0$, is not possible to guarantee the absolute stability globally, but we must reduce the domain to a finite interval and guarantee the absolute stability in this interval. Let us choose $\beta = 1$ in order to include the linear part of the saturation, and then let us choose the circle for α in order to enlarge the interval in which the system is absolutely stable (see Fig. 1.11). By choosing $\alpha = 0.54$ one has $a = 1/\alpha = 1.85$ and the sector condition holds in the interval $[-1.85, 1.85]$. Therefore, the system is absolutely stable if the Nyquist plot of $G(j\omega)$ encircles the disk $\mathcal{C}(-1/\alpha, -1/\beta) = \mathcal{C}(-1.85, -1)$ once counterclockwise, as reported in Fig. 1.11. Notice that, enlarging α reduces both the domain and the sector condition of the nonlinearity; on the contrary, choosing α small (close to 0) leads to a big disk which is difficult to be encircled. $\triangle$

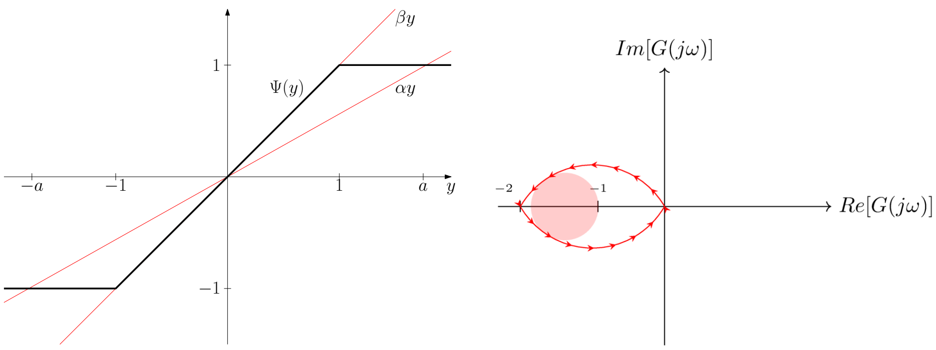


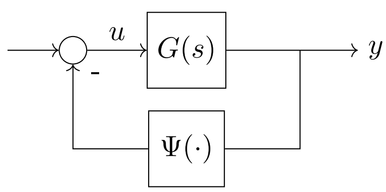
Figure 1.11: Sector condition and Nyquist plot of Example 1.6.

1.4 Popov Criterion

If the nonlinearity Ψ is time invariant, which means $\Psi(t, y) = \Psi(y)$, we can use a different Lyapunov function (Lure type Lyapunov function), which allows one to derive the Popov criterion.

$$V(x) = x^T P x + 2\eta \int_0^y \Psi^T(\sigma) K d\sigma \tag{1.11}$$

with $\eta \geq 0$. Let us consider again the following feedback system:



$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \\ u = -\Psi(y) \\ G(s) = C(sI - A)^{-1}B \end{cases} \tag{1.12}$$

As for the circle criterion, assume

- $x \in \mathbb{R}^n, \quad y \in \mathbb{R}^p$
- (A,B) controllable
- (A,C) observable

Suppose the nonlinearity is sector bounded and assume we have already applied the pole shifting procedure, in order to obtain

$$\begin{aligned} & G \in \mathcal{H} \\ & \Psi^T(y)[\Psi(y) - Ky] \leq 0, \;\; K > 0, \;\; \forall y \in \Gamma \subseteq \mathbb{R}^p \end{aligned} \tag{1.13}$$

A further assumption is that $K\Psi$ be the gradient of a scalar function and

$$\int_0^y \Psi^T(\sigma) K d\sigma \geq 0, \;\; \forall y \in \Gamma \subseteq \mathbb{R}^p \tag{1.14}$$

These conditions are satisfied, for instance, when $K = cI$ ($c > 0$ constant), the Jacobian matrix $\frac{\partial \Psi}{\partial y}$ is symmetric and $\int_0^y \Psi^T(\sigma) d\sigma \geq 0$. In the SISO case, the nonlinearity should be such that its integral is always greater than zero, because the Lyapunov function must be positive defined (see Fig. 1.12).

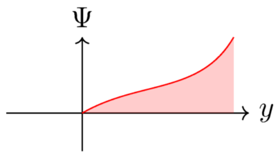


Figure 1.12: Integral of Ψ in the SISO case.

Let us calculate the gradient of the Lure type Lyapunov function (1.11).

$$\begin{aligned} \dot{V}(x) &= x^T(A^TP + PA)x - 2x^TPB\Psi + 2\eta\Psi^T K \overbrace{C}^{\frac{\partial y}{\partial x}} \overbrace{(Ax - B\Psi)}^{\frac{\partial x}{\partial t}} \\ &\leq x^T(A^TP + PA)x - 2x^TPB\Psi + 2\eta\Psi^TKCAx - 2\eta\Psi^TKCB\Psi \overbrace{-2\Psi^T\Psi + 2\Psi^TKy}^{-2\Psi^T[\Psi - Ky] \geq 0} \\ &= x^T(A^TP + PA)x - 2x^TPB\Psi + 2\eta\Psi^TKCAx + 2\Psi^TK\overbrace{Cx}^{y} - 2\eta\Psi^TKCB\Psi - 2\Psi^T\Psi \\ &= x^T(A^TP + PA)x - 2x^T[PB - \eta A^TC^TK - C^TK]\Psi - \Psi^T[2I + 2\eta KCB]\Psi \\ &= x^T(A^TP + PA)x - 2x^T[PB - \eta A^TC^TK - C^TK]\Psi - \Psi^T[2I + \eta KCB + \eta B^TC^TK]\Psi \end{aligned}$$

In the last equality we exploit the property that $\Psi^TQx = x^TQ^T\Psi$ for any $p \times p$ matrix Q.

Now, let us choose η and matrix W such that

$$W^TW = 2I + \eta KCB + \eta B^TC^TK \geq 0$$

It can be proved that such η and W always exist.

Suppose that

$$\exists\, P = P^T > 0, L, \epsilon \begin{cases} A^TP + PA = -L^TL - \epsilon P \\ PB = C^TK + \eta A^TC^TK - L^TW \end{cases} \tag{1.15}$$

One has,

$$\begin{aligned} \dot{V}(x) &\leq -\epsilon x^TPx - x^TL^TLx + 2x^TL^TW\Psi - \Psi^TW^TW\Psi = \\ &= -\epsilon x^TPx - (x^TL^TLx - 2x^TL^TW\Psi + \Psi^TW^TW\Psi) = \\ &= -\epsilon x^TPx - ([Lx - W\Psi]^T[Lx - W\Psi]) \leq -\epsilon x^TPx \end{aligned}$$

which means that $\dot{V}(x) < 0,\ \forall x \in \mathbb{R}^n \backslash \{0\}$. So, $\dot{V}$ is negative defined and therefore the system is globally and asymptotically stable.

It remains to find when it is possible to choose P, L and ϵ as in (1.15). To this purpose, we will make use of the KYP Lemma 1.2 that we report here for convenience.

Let

$$Z(s) = \mathcal{C}(sI - \mathcal{A})^{-1}\mathcal{B} + \mathcal{D}$$

with $Z(s) \in \mathbb{C}^{p\times p}$ and $\mathcal{A} \in \mathcal{H}$ (Hurwitz), $(\mathcal{A}, \mathcal{B})$ controllable, $(\mathcal{A}, \mathcal{C})$ observable. Then, $Z(s)$ is SPR if and only if $\exists P = P^T > 0,\ L,\ W,\ \epsilon$, such that

- $P\mathcal{A} + \mathcal{A}^T P = -L^T L - \epsilon P$
- $P\mathcal{B} = \mathcal{C}^T - L^T W$
- $\mathcal{D} + \mathcal{D}^T = W^T W$

So, we can make a parallel between our function and the generic function of the KYP lemma, and try to choose our parameters such that they satisfy the constrains on $\dot{V}$ and also the condition for SPR dictated by Lemma 1.2. In particular, by choosing

- $\mathcal{A} = A$
- $\mathcal{B} = B$
- $\mathcal{C} = KC + \eta KCA$
- $\mathcal{D} = I + \eta KCB$

we find that

$$Z(s) = I + \eta KCB + (KC + \eta KCA)(sI - A)^{-1}B \tag{1.16}$$

which by Lemma 1.2 is SPR. Since $G(s) = C(sI - A)^{-1}B$, we can rewrite (1.16) as

$$\begin{aligned}
Z(s) &= I + \eta KCB + KC(sI - A)^{-1}B + \eta KCA(sI - A)^{-1}B = \\
&= I + \eta KC[I + A(sI - A)^{-1}]B + KC(sI - A)^{-1}B = \\
&= I + \eta KC[(sI - A)(sI - A)^{-1} + A(sI - A)^{-1}]B + KC(sI - A)^{-1}B = \\
&= I + \eta KC[(sI - A) + A)](sI - A)^{-1}B + KC(sI - A)^{-1}B = \\
&= I + \eta KC(sI)(sI - A)^{-1}B + KC(sI - A)^{-1}B = \\
&= I + \eta sKG + KG = \\
&= I + (1 + \eta s)KG \text{ is SPR}
\end{aligned}$$

Here, all variables are known except η. We must choose it such that $W^T W > 0$. Moreover, to preserve the observability of $(\mathcal{A}, \mathcal{C})$, $-1/\eta$ cannot be an eigenvalue of A. In fact, if $\eta = -1/\lambda_i$ where λ_i is an eigenvalue of A with associated eigenvector v_i, one has

$$\mathcal{C}v_i = (KC + \eta KCA)v_i = KC(I + \eta A)v_i = KC(1 + \eta\lambda_i)v_i = 0$$

and hence $(\mathcal{A}, \mathcal{C})$ is not observable.

Note. If $G(s) \in \mathcal{H}$, by choosing $\eta = 0$ we obtain the circle criterion. So, by choosing $\eta > 0$, the Popov criterion provides less conservative conditions with respect to the circle criterion.

Let us summarize the multivariable Popov criterion.

Theorem 1.5 (Multivariable Popov criterion)**.** *Let the system* (1.12) *be given, where* $A \in \mathcal{H}$*,* (A, B) *is controllable,* (A, C) *is observable, and* Ψ *is a time-invariant nonlinearity satisfies the sector condition* (1.13)*. Assume* $K\Psi$ *is the gradient of a scalar function and* (1.14) *holds. Then, the system is absolutely stable if there exists* $\eta \geq 0$ *with* $-1/\eta$ *not an eigenvalue of* A *such that*

$$Z(s) = I + (1 + \eta s)KG(s)$$

is SPR.

1.4.1 Popov criterion for SISO systems

Let us apply the Popov criterion to the SISO case. It is required that

$$Z(s) = 1 + \underbrace{(1 + \eta s)}_{\text{multiplicator}} kG(s) \text{ is SPR}$$

Since $G \in \mathcal{H}$ also $Z \in \mathcal{H}$. Moreover, choose η such that $Z(\infty) > 0$.

To guarantee SPR, we must verify that

$$\text{Re}[1 + (1 + \eta j\omega)kG(j\omega)] > 0, \ \forall\omega \tag{1.17}$$

Let us define $G(j\omega) = x(j\omega) + jy(j\omega)$. So, (1.17) becomes

$$\begin{aligned}
&1 + \text{Re}[(1 + \eta j\omega)k(x(j\omega) + jy(j\omega))] > 0\\
&\frac{1}{k} + \text{Re}[x(j\omega) + jy(j\omega) + j\omega\eta x(j\omega) - \omega\eta y(j\omega)] > 0\\
&\frac{1}{k} + x(j\omega) - \eta\omega y(j\omega) > 0
\end{aligned}$$

By defining $y'(j\omega) \equiv \omega y(j\omega)$, one gets

$$\begin{aligned}
\frac{1}{k} + x(j\omega) - \eta y'(j\omega) > 0&\\
\eta y'(j\omega) &< x(j\omega) + \frac{1}{k}\\
y'(j\omega) &< \frac{1}{\eta}x(j\omega) + \frac{1}{k\eta}
\end{aligned}$$

This conditions for absolute stability leads to the *Popov diagram*, in which the vertical axis is not the imaginary axis, but the y' axis (i.e., the imaginary axis multiplied by ω). In conclusion, the system is absolutely stable if the Popov diagram lies to the right of the mentioned line, as shown in Fig. 1.13.

Example 1.7. Find a sector condition for the nonlinearity Ψ (i.e., find α and β) such that the following system is absolutely stable.

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_2 + u \\ y = x_1 \\ u = -\Psi(y) \end{cases} \qquad \begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y = [1 \quad 0]\, x \end{cases}$$

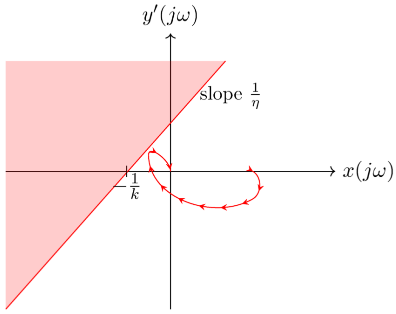
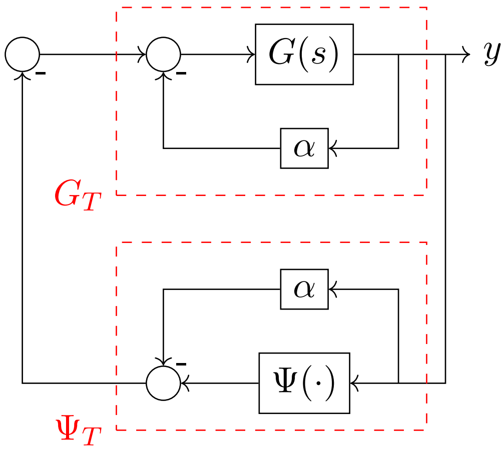


Figure 1.13: Popov diagram.

Notice that $G(s) \notin \mathcal{H}$. In fact, an eigenvalue of the matrix A is 0, or equivalently the corresponding transfer function has a pole in 0.

$$G(s) = C(sI - A)^{-1}B = \begin{bmatrix} 1 & 0 \end{bmatrix} \left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \right) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{s(s+1)}$$

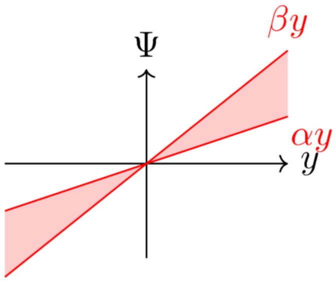
We apply the pole shifting technique as shown in the following scheme.



One obtains

$$\begin{cases} G_T = \dfrac{G}{1+\alpha G} = \dfrac{\frac{1}{s(s+1)}}{1 + \frac{\alpha}{s(s+1)}} = \dfrac{1}{s^2 + s + \alpha} \\ \Psi_T(y) = \Psi(y) - \alpha y \end{cases}$$

Since $\alpha > 0$, it holds $G_T \in \mathcal{H}$. Moreover, the sector $[\alpha, \beta]$ becomes $[0, k]$, where $k = \beta - \alpha$. Therefore, we may apply the Popov criterion even if the original sector condition is of the form $\beta > \alpha > 0$, as in the following figure.



Let $Z(s) = 1 + (1 + \eta s)kG_T(s)$. It is easy to see that $Z(\infty) = 1$ for any η. So, it remains to study

$$\begin{aligned}\frac{1}{k} + \mathrm{Re}\left[(1 + j\omega\eta)G_T(j\omega)\right] =& \frac{1}{k} + \mathrm{Re}\left[(1 + j\omega\eta)\left(\frac{1}{\alpha - \omega^2 + j\omega}\right)\right] > 0\\ & \frac{1}{k} + \mathrm{Re}\left[(1 + j\omega\eta)\left(\frac{\alpha - \omega^2 - j\omega}{(\alpha - \omega^2)^2 + \omega^2}\right)\right] > 0\\ & \frac{1}{k} + \frac{\alpha - \omega^2}{(\alpha - \omega^2)^2 + \omega^2} + \frac{\eta\omega^2}{(\alpha - \omega^2)^2 + \omega^2} > 0\end{aligned}$$

which is satisfied for $\eta \geq 1$. Let us draw the Popov diagram, remembering that

$$\begin{cases} x = \mathrm{Re}[G_T(j\omega)] \\ y' = \omega\,\mathrm{Im}[G_T(j\omega)] \end{cases}$$

In this case, with $\eta = 1$ the system is absolutely stable for every k (also $k \to \infty$), i.e., also for $\beta \to \infty$ and $\alpha \to 0$. △

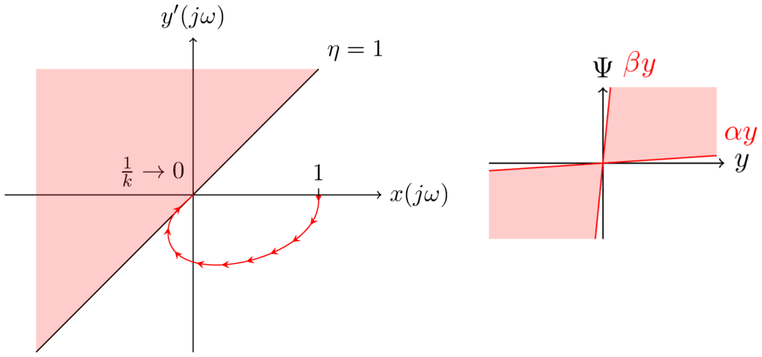


Figure 1.14: Popov diagram of Example 1.7 for $\eta = 1$.